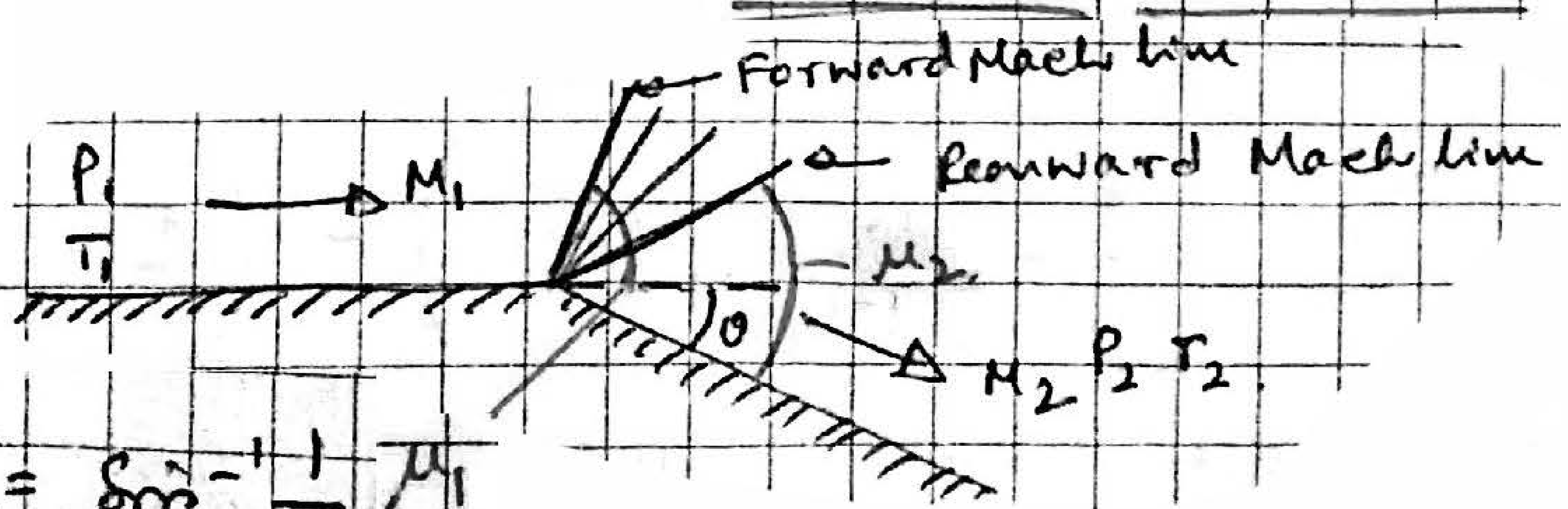
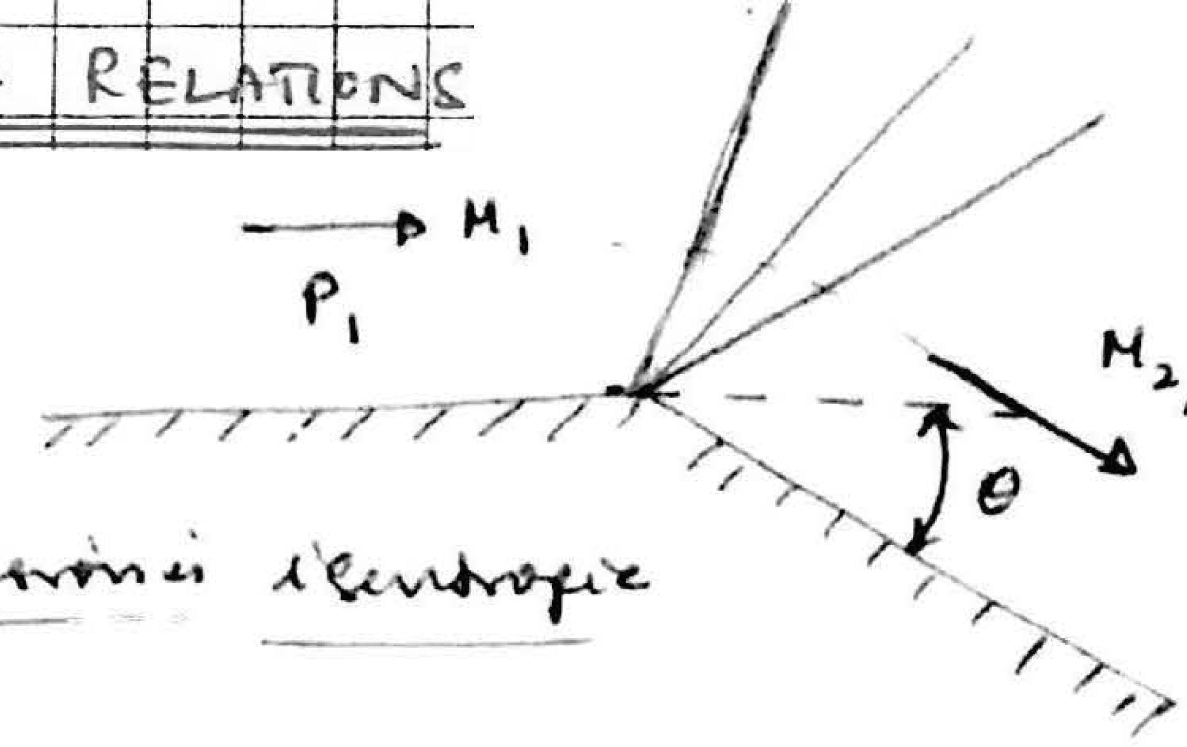
HYPERSONIC EXPANSION WAVE RELATIONS





Expansion is isentropic

$\mu = \sin^{-1}\frac{1}{M}$ μ_1

Mach angle $\mu_1 = \sin^{-1}\frac{1}{M_1}$ $\mu_2 = \sin^{-1}\frac{1}{M_2}$

$M_1, P_1 \rightarrow$ upstream properties.

$M_2, P_2 \rightarrow$ downstream properties.

From basic compressible flow, the relation between θ, M_1 and M_2 is given by

$$\boxed{\theta = \nu(M_2) - \nu(M_1)} \qquad (1)$$

$\nu \rightarrow$ Prandtl–Meyer function.

Refer to a book on gas dynamics.

$$\boxed{\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \times \left[\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)}\right] - \tan^{-1}\sqrt{M^2-1}} \qquad (2)$$

For large Mach numbers, $\sqrt{M^2-1} \approx M$.

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left[\sqrt{\frac{\gamma-1}{\gamma+1}}\,M\right] - \tan^{-1}M \qquad (3)$$

$$\tan^{-1}x = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x}\right)$$

From the series expansion,

$$(-1 < x < 1)$$

$$\tan^{-1}\left(\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{3}\frac{1}{x^3} + \frac{1}{5}\frac{1}{x^5} - \frac{1}{7}\frac{1}{x^7} + \dots$$

$$\dots\dots$$

Hence, from the above, we get.

$$\tan^{-1}x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots\dots - \qquad (4)$$

Using (4) to expand equation (3) we get

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\left(\frac{\pi}{2} - \sqrt{\frac{\gamma+1}{\gamma-1}}\frac{1}{M} + \dots\right) - \left(\frac{\pi}{2} - \frac{1}{M} + \dots\right) \qquad (5)$$

The higher order terms, including $\frac{1}{3M^3}$, $\frac{1}{5M^5}$ etc. can be ignored at high Mach numbers.
(hypersonic)

Hence, $\boxed{\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\frac{\pi}{2} - \frac{\gamma+1}{\gamma-1}\frac{1}{M} - \frac{\pi}{2} + \frac{1}{M}}$ — (6)

can be written as

Substituting equation (6) into equation (1), we obtain,

$$\theta = \nu(M_2) - \nu(M_1)$$

$$\theta = \frac{1}{M_2} - \left(\frac{\gamma+1}{\gamma-1}\right)\frac{1}{M_2} - \frac{1}{M_1} + \left(\frac{\gamma+1}{\gamma-1}\right)\frac{1}{M_1}$$

$$\theta = \frac{1}{M_2}\left[1 - \frac{\gamma+1}{\gamma-1}\right] - \frac{1}{M_1}\left[1 - \frac{\gamma+1}{\gamma-1}\right]$$

$$= \frac{1}{M_2}\left[-\frac{2}{\gamma-1}\right] - \frac{1}{M_1}\left[-\frac{2}{\gamma-1}\right]$$

$$\boxed{\theta = \frac{2}{\gamma-1}\left[\frac{1}{M_1} - \frac{1}{M_2}\right]} \quad - (7)$$

Eqn. 12 is a hypersonic relation for Prandtl-Meyer expansion waves. As M_1 and M_2 become larger, it becomes more accurate. M_2 is always larger than M_1 in an expansion, and hence θ is a +ve quantity.

The expansion through a fan is isentropic. Hence isentropic relations can be used:

$$h_0 = h + \frac{v^2}{2}$$

$$c_p T_0 = c_p T + \frac{v^2}{2}$$

$$c_p T_0 - c_p T = \frac{v^2}{2}$$

$$c_p (T_0 - T) = \frac{v^2}{2}$$

$$T_0 - T = \frac{v^2}{2c_p} = \frac{v^2}{2\frac{\gamma R}{\gamma-1}}$$

$$c_p = \frac{\gamma R}{\gamma-1}$$

$$T_0 - T = \frac{(\gamma-1)}{2}\frac{v^2}{\gamma R}$$

$$\frac{T_0}{T} - 1 = \frac{\gamma-1}{2}\frac{v^2}{\gamma R T}$$

$$\frac{T_0}{T} = \frac{\gamma-1}{2}\frac{v^2}{a^2} + 1$$

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2$$

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/\gamma-1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

$$h_0 = h_1 + \frac{v_1^2}{2}$$

$$\frac{T_0}{T} = \left[1 + \frac{\gamma-1}{2}M^2\right] \Rightarrow \frac{T_0}{T_1} = \left[1 + \frac{\gamma-1}{2}M_1^2\right]$$

$$\frac{T_0}{T_1} \times \frac{T_2}{T_0} = \frac{T_2}{T_1} = \left[1 + \frac{\gamma-1}{2}M_1^2\right] \times \left[\frac{1}{1+\frac{\gamma-1}{2}M_2^2}\right]$$

$$\frac{T_2}{T_1} = \left[\frac{1 + \frac{(\gamma-1)}{2}M_1^2}{1+\frac{(\gamma-1)}{2}M_2^2}\right]$$

or $$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} \Rightarrow \frac{p_2}{p_1} = \left[\frac{1+\frac{(\gamma-1)}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2}\right]^{\frac{\gamma}{\gamma-1}} \quad -(8)$$

Now at hypersonic Mach numbers $M_1 >>> 1$ and M_2 will be still larger

$$\frac{p_2}{p_1} = \left[\frac{\frac{\gamma-1}{\cancel{2}}M_1^2}{\frac{\gamma-1}{\cancel{2}}M_2^2}\right]^{\frac{\gamma}{(\gamma-1)}}$$

$$\frac{p_2}{p_1} = \left[\frac{M_1}{M_2}\right]^{\frac{2\gamma}{\gamma-1}} \quad -(9)$$

$$\theta = \frac{2}{(\gamma-1)}\left[\frac{1}{M_1} - \frac{1}{M_2}\right]$$

multiply and divide by M_1 on the right hand side

$$\theta = \frac{2}{(\gamma-1)}\left[\frac{1}{M_1} - \frac{1}{M_2}\right]\frac{M_1}{M_1} = \frac{2}{\gamma-1}\left[\frac{M_1}{M_1^2} - \frac{M_1}{M_1M_2}\right]$$

$$\frac{\theta(\gamma-1)}{2} = \frac{M_{\cancel{1}}}{M_1^{\cancel{2}}} - \frac{M_1}{M_2} \times \frac{1}{M_1}$$

$$= \frac{1}{M_1} - \frac{M_1}{M_2} \times \frac{1}{M_1}$$

$$\frac{\theta(\gamma-1)}{2} - \frac{1}{M_1} = -\frac{M_1}{M_2} \times \frac{1}{M_1}$$

$$\frac{1}{M_1} - \frac{\theta(\gamma-1)}{2} = \frac{M_1}{M_2} \times \frac{1}{M_1}$$

$$\frac{\cancel{M_1}}{\cancel{M_1}} - \frac{\theta(\gamma-1)}{2}M_1 = \frac{M_1}{M_2}.$$

$$\boxed{1 - \frac{1}{2}\theta(\gamma-1)M_1 = \frac{M_1}{M_2}} \quad -(10)$$

Substitute eqn (10) into (9)

$$\frac{p_2}{p_1} = \left[1 - \frac{(\gamma-1)}{2} M_1\theta\right]^{\frac{2\gamma}{\gamma-1}}$$

$$\boxed{\frac{p_2}{p_1} = \left[1 - \frac{(\gamma-1)}{2} K\right]^{\frac{2\gamma}{(\gamma-1)}}} \quad -(11)$$

The above eqn is the pressure ratio across the expansion wave at hypersonic Mach numbers. It is in terms of hypersonic similarity parameter $(K = M_1\theta)$. Using the above eqn, just by knowing the upstream Mach number M_1 and the flow deflection angle θ, we can get the Pressure ratio across an expansion corner.

coefficient of pressure $$C_p = \frac{p_2 - p_1}{q_1} = \frac{p_2 - p_1}{\frac{\gamma}{2} p_1 M_1^2} = \frac{2}{\gamma M_1^2}\left[\frac{p_2}{p_1} - 1\right]$$

Substituting for $\frac{p_2}{p_1}$ from (11) we get

$$C_p = \frac{2}{\gamma M_1^2}\left\{\left[1 - \frac{(\gamma-1)}{2} K\right]^{\frac{2\gamma}{(\gamma-1)}} - 1\right\}$$

R.H.S $\times \div$ by θ^2 [multiply and divide by θ^2]

$$C_p = \frac{2\theta^2}{\gamma M_1^2 \theta^2}\left\{\left[1 - \frac{(\gamma-1)}{2} K\right]^{\frac{2\gamma}{(\gamma-1)}} - 1\right\}$$

$$\boxed{C_p = \frac{2\theta^2}{\gamma K^2}\left\{\left[1 - \frac{(\gamma-1)}{2} K\right]^{\frac{2\gamma}{(\gamma-1)}} - 1\right\}} \quad -(12)$$

Equation (12) for the hypersonic expansion wave is analogous to the C_p equation for hypersonic shock wave. in the sense

$$\boxed{\frac{C_p}{\theta^2} = f(K, \gamma)} \quad -(13)$$