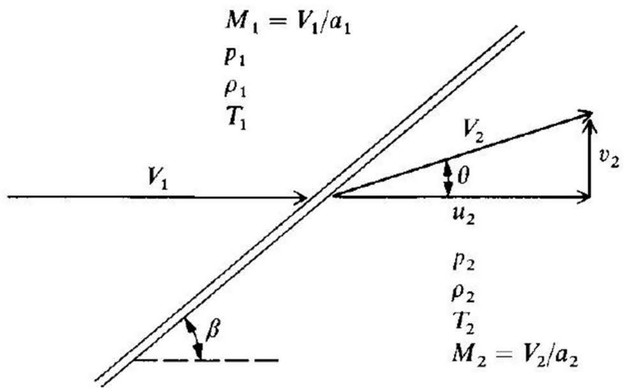
Hypersonic Shock and Expansion-Wave Relations



Exact:

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 \sin^2 \beta - 1)$$

as M_1 goes to infinity $\qquad M_1^2 \sin^2 \beta \gg 1$

$$\boxed{\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1} M_1^2 \sin^2 \beta}$$

Exact:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2 \sin^2 \beta}{(\gamma-1)M_1^2 \sin^2 \beta + 2}$$

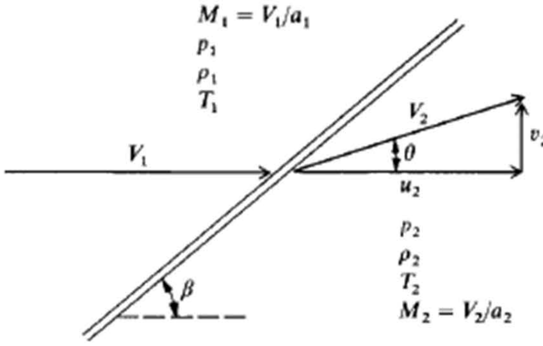
As $M_1 \to \infty$:

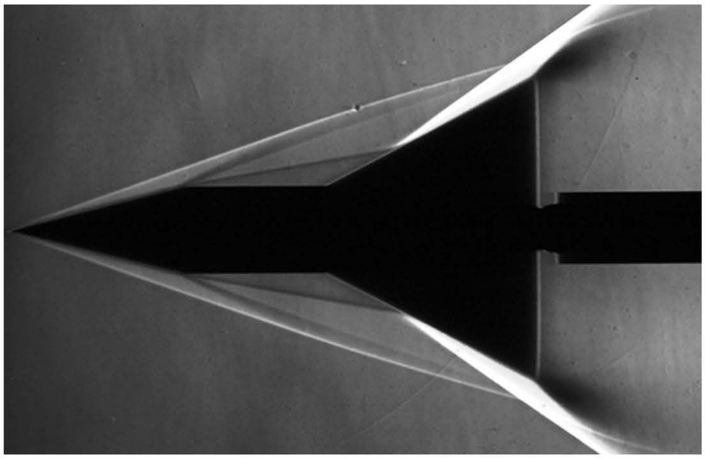
$$\boxed{\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}}$$

$$\frac{T_2}{T_1} = \frac{(p_2/p_1)}{(\rho_2/\rho_1)} \text{(from the equation of state } p = \rho RT)$$

As $M_1 \to \infty$:

$$\boxed{\frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} M_1^2 \sin^2 \beta}$$





$$\frac{u_2}{V_1} = 1 - \frac{2(M_1^2 \sin^2 \beta - 1)}{(\gamma + 1)M_1^2} \qquad \frac{v_2}{V_1} = \frac{2(M_1^2 \sin^2 \beta - 1)\cos\beta}{(\gamma+1)M_1^2}$$

↑ optional

coefficient of pressure.

In aerodynamics, pressure distributions over a body ~~(in general)~~ are quoted in terms of coefficient of pressure.

$$C_p = \frac{p_2 - p_1}{q_1} \qquad \left[\frac{p - p_\infty}{q_\infty} \right]$$

q_1 = is the upstream dynamic pressure [which is only a definition for high speed flows]

$$q_1 = \frac{\gamma}{2} p_\infty M_\infty^2 \qquad q_1 = \frac{1}{2}\rho_1 V_1^2 \qquad M_1 = \frac{V_1}{a_1}$$

$$q_1 = \frac{\gamma}{2} p_1 M_1^2 \qquad = \frac{1}{2}\rho_1 M_1^2 a_1^2 \qquad p = \rho R T$$

$$C_p = \frac{p_2 - p_1}{\frac{\gamma}{2} p_1 M_1^2} = \frac{2}{\gamma M_1^2}\left[\frac{p_2}{p_1} - 1\right] \qquad = \frac{1}{2}\rho_1 M_1^2 \gamma R T_1$$

$$= \frac{1}{2} M_1^2 \gamma p_1$$

$$C_p = \frac{2}{\gamma M_1^2}\left[\frac{p_2}{p_1} - 1\right] \qquad = \frac{\gamma}{2} p_1 M_1^2$$

substituting for $\frac{p_2}{p_1}$

$$C_p = \frac{2}{\gamma M_1^2}\left[1 + \frac{2\gamma}{\gamma+1}(M_1^2 \sin^2\beta - 1)\right]$$

when $M_1 \gg 1$

$$= \frac{2}{\gamma M_1^2} + \frac{4\gamma}{\gamma M_1^2(\gamma+1)}(M_1^2 \sin^2\beta - 1) \Rightarrow \cancel{\frac{2}{\gamma M_1^2}} + \frac{4\gamma M_1^2 \sin^2\beta}{\gamma M_1^2(\gamma+1)} - \cancel{\frac{4\gamma}{\gamma M_1^2(\gamma+1)}}$$

$$C_p = \frac{4 \sin^2\beta}{(\gamma+1)}$$

or $$\boxed{C_p = \frac{4}{\gamma+1}(\sin^2\beta)}$$

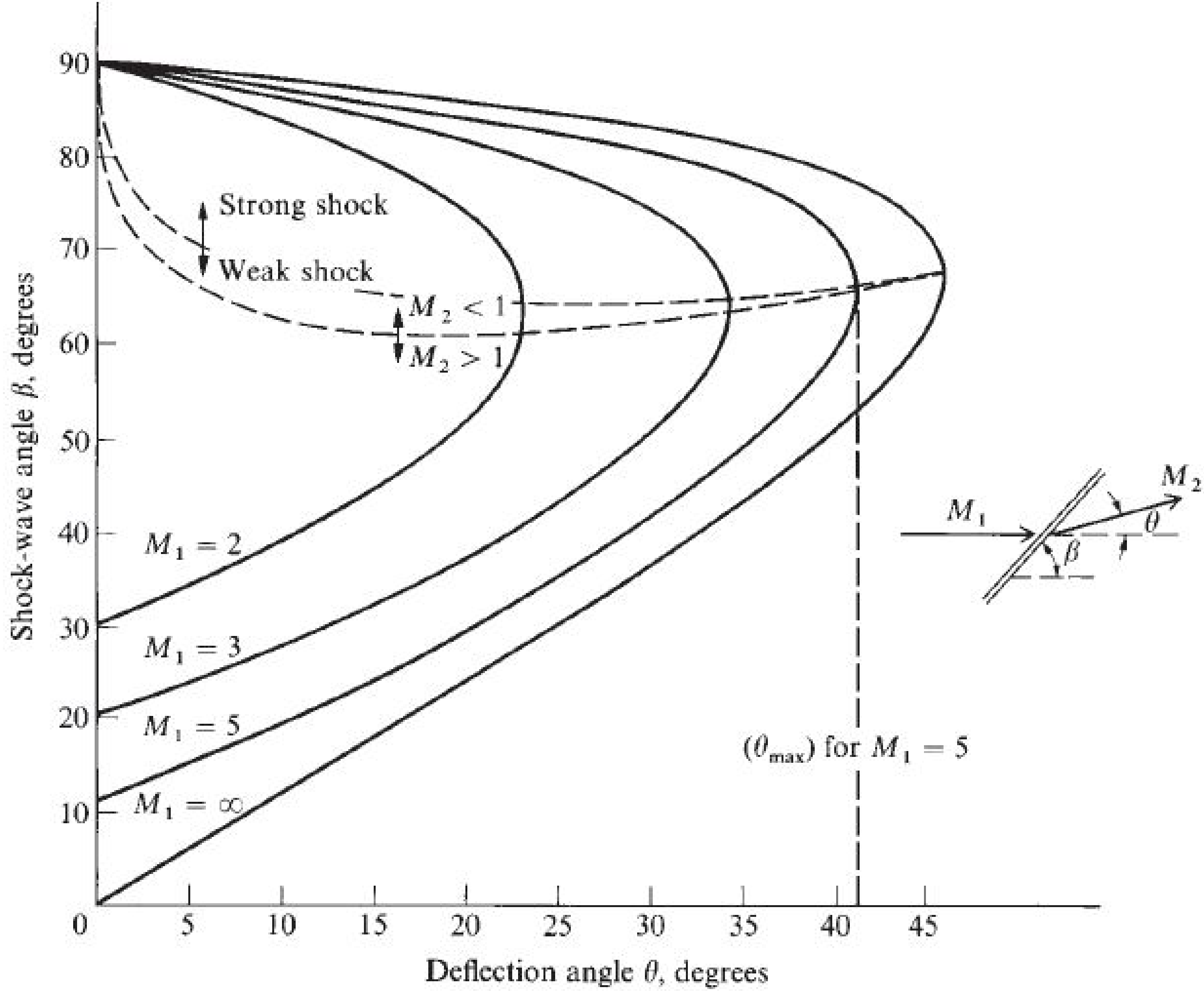


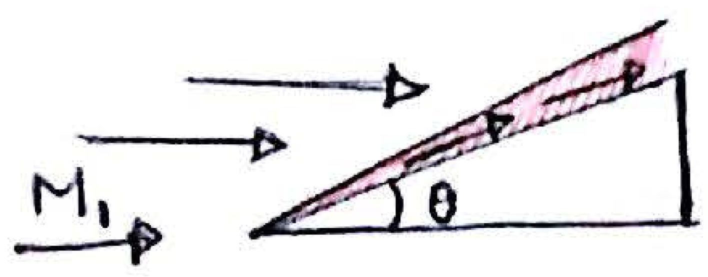
Fig. 2.3 ***θ-β-M*** **diagram.**

In case of pressure/temperature ratio $\frac{P_2}{P_1}$ (T_2/T_1); it is easy to express the relation in terms of 'K' hypersonic similarity parameter.

The usefulness can be demonstrated in the easy the θ, β, M relation and, in terms of getting an expression $\beta = \beta(\theta)$.

Consider the θ, β, M relation $\rightarrow \theta = \theta(\beta)$

$\beta = \beta(\theta) \rightarrow$ is very difficult to obtain.



θ, β, M relation rewritten below

$\gamma = 1.4$ Air. $\beta = 1.2\theta$

Only for small angles. (say up to 15°)

$$\boxed{\tan\theta = 2\cot\beta \left[\frac{M_1^2 \sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}\right]}$$

① For small θ and β.

$$(\cancel{\tan})\ \theta = \frac{2}{\beta}\left[\frac{M_1^2\beta^2 - 1}{M_1^2(\gamma+1)+2}\right]$$

$$\theta\beta = 2\left[\frac{M_1^2\beta^2 - 1}{M_1^2(\gamma+1)+2}\right]$$

$$M_1^2\beta^2 - 1 = \left[\frac{M_1^2(\gamma+1)+2}{2}\right]\theta\beta.$$

$$M_1^2\beta^2 - 1 = \left[\frac{M_1^2(\gamma+1)}{2} + 1\right]\beta\theta$$

② Now Apply the high Mach number limit but $\boxed{\text{Say } M_1 \text{ is finite.}}$

$$M_1^2\beta^2 - 1 = \left[\frac{M_1^2(\gamma+1)}{2}\right]\beta\theta$$

$M_1^2\beta^2$ is not very large in comparison with unity as β is small.

$$M_1^2\beta^2 - 1 = \frac{\gamma+1}{2} M_1^2\beta\theta$$

Re-arranging, we obtain

$(\div M_1^2\theta^2)$

$$\frac{\cancel{M_1^2}\beta^2}{\cancel{M_1^2}\theta^2} - \frac{1}{M_1^2\theta^2} = \frac{\gamma+1}{2}\frac{\cancel{M_1^2}\beta\cancel{\theta}}{\cancel{M_1^2}\theta^{\cancel{2}}}$$

$$a\ \left(\frac{\beta}{\theta}\right)^2 - \overset{b}{\frac{\gamma+1}{2}}\left(\frac{\beta}{\theta}\right) - \frac{1}{M_1^2\theta^2} = 0$$

This is a quadratic eqn of the form $ax^2+bx+c=0$, the soln for which is $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ ruled out using this and re-arranging, we get

$$\boxed{\frac{\beta}{\theta} = \frac{\gamma+1}{4} + \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}}}$$

This eqn is an explicit relation for $\beta = \beta(\theta)$, valid for small angles and high but finite Mach numbers ~~(but)~~.

Now consider the equation for pressure ratio across an oblique shockwave

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left[M_1^2\sin^2\beta - 1\right]$$

② For small angles,

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left[M_1^2\beta^2 - 1\right]$$

③ Hypersonic, but finite Mach number,

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left[M_1^2\beta^2 - 1\right] \qquad \because \beta \text{ is small here } M_1^2\beta^2 \cancel{\gg} 1$$

The above expression is still in terms of β.

By using the eqn $\displaystyle \frac{\beta}{\theta} = \frac{\gamma+1}{4} + \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}}$ $\qquad \beta = \beta(\theta)$

$$\left(\frac{\beta}{\theta}\right)^2 = \left(\frac{\gamma+1}{4}\right)^2 + \left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2} + 2\,\frac{\gamma+1}{4}\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}}$$

$$\left(\frac{\beta}{\theta}\right)^2 = \frac{(\gamma+1)^2}{16} + \frac{(\gamma+1)^2}{16} + \frac{1}{M_1^2\theta^2} + \frac{\gamma+1}{2}\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}}$$

or

$$\beta^2 = \left[\frac{(\gamma+1)}{2}\frac{(\gamma+1)}{4} + \frac{\gamma+1}{2}\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}}\right]\theta^2 + \frac{1}{M_1^2} \quad -①$$

Substituting, eqn ① above into the eqn for $\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left[M_1^2\beta^2 - 1\right]$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left[M_1^2\theta^2\frac{(\gamma+1)}{2}\frac{(\gamma+1)}{4} + \theta^2 M_1^2\frac{(\gamma+1)}{2}\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}} + \cancel{1} - \cancel{1}\right]$$

$$= 1 + \frac{2\gamma}{\gamma+1}\left[M_1^2\theta^2\frac{(\gamma+1)^2}{8} + \frac{(\gamma+1)}{2}\cdot M_1^2\theta^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}} \cancel{+1-1}\right]$$

$$= 1 + \frac{\cancel{2}\gamma}{\cancel{(\gamma+1)}} M_1^2\theta^2\frac{(\gamma+1)^{\cancel{2}}}{\cancel{8}\,4} + \frac{\cancel{2}\gamma}{\cancel{\gamma+1}} \times \frac{\cancel{\gamma+1}}{\cancel{2}} \times M_1^2\theta^2 \times \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}}$$

$$\boxed{\frac{p_2}{p_1} = 1 + \frac{\gamma}{4}M_1^2\theta^2(\gamma+1) + \gamma M_1^2\theta^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}}}$$

$$\boxed{\frac{P_2}{P_1} = 1 + \frac{\gamma(\gamma+1)}{4} M_1^2\theta^2 + \gamma\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_1^2\theta^2}}\cdot M_1^2\theta^2} \quad -(2)$$

$K \equiv M_1\theta$ using this in the above relation,

$$\boxed{\frac{P_2}{P_1} = 1 + \frac{\gamma(\gamma+1)}{4} K^2 + \gamma\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}}\cdot K^2} \quad -(3)$$

$$\frac{P_2}{P_1} = \frac{P_2}{P_1}(K)$$

The above relation can be very useful when the flow deflection angles are small, and the Mach number is large, but finite, and it gives the pr. ratio as a function of K which is nothing but $M_1 \times \theta$, both of which are generally known

Revisit the eqn. for coefficient of pressure; $C_p = \frac{P_2 - P_1}{q_1} = \frac{P_2 - P_1}{\frac{\gamma}{2} P_1 M_1^2}$

$$C_p = \frac{2}{\gamma M_1^2}\left[\frac{P_2}{P_1} - 1\right]$$

Substitute equation (3) in the above for C_p;

we get

$$C_p = \frac{\cancel{2}}{\cancel{\gamma} M_1^2}\left[\cancel{1} + \frac{\cancel{\gamma}(\gamma+1)}{\cancel{4}_2} K^2 + \cancel{\gamma} K^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}} - \cancel{1}\right]$$

$$C_p = \frac{\gamma+1}{2M_1^2} K^2 + \frac{2K^2}{M_1^2}\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}}$$

$$C_p = \frac{\gamma+1}{2\cancel{M_1^2}} \times \cancel{M_1^2}\theta^2 + \frac{2\cancel{M_1^2}\theta^2}{\cancel{M_1^2}}\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}}$$

$$C_p = \frac{\gamma+1}{2}\theta^2 + 2\theta^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}} = \boxed{2\theta^2\left[\frac{\gamma+1}{4} + \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}}\right] = C_p}$$

(4)

For hypersonic flow over wedges with small deflection angles,

$$\boxed{\frac{C_p}{\theta^2} = f(\gamma, K)} \quad -(5)$$

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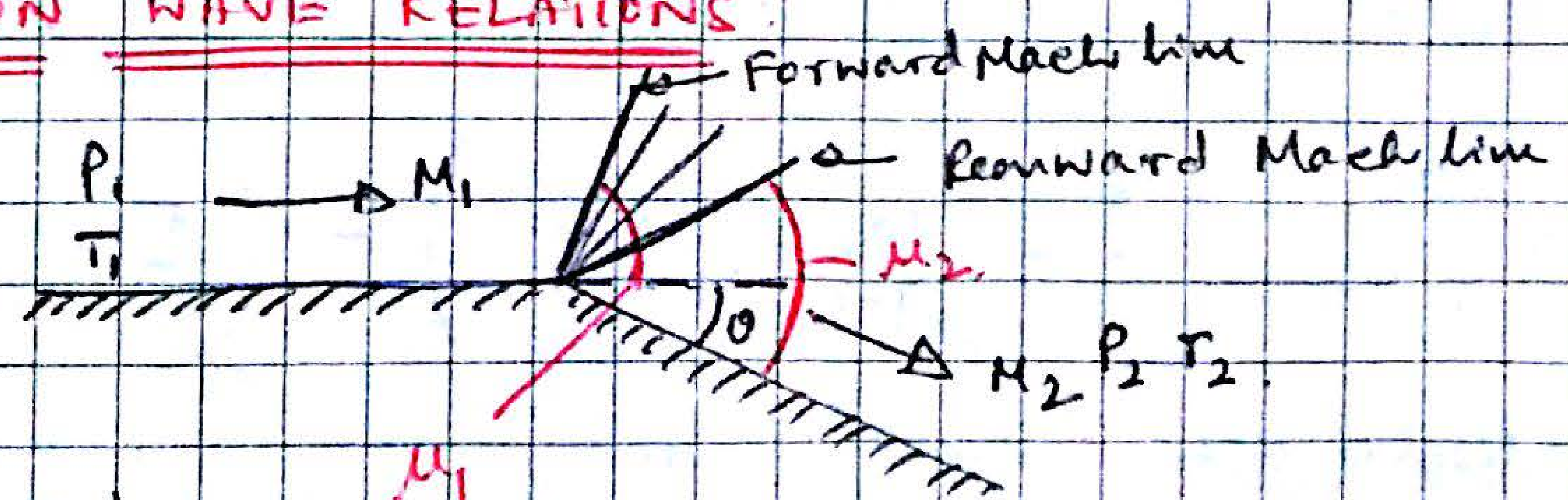
HYPERSONIC EXPANSION WAVE RELATIONS

$\mu = \sin^{-1}\frac{1}{M}$

Mach angle



$\mu_1 = \sin^{-1}\frac{1}{M_1}$

$\mu_2 = \sin^{-1}\frac{1}{M_2}$

P.T.O