SECONDARY FLOWS: THEORY, EXPERIMENT, AND APPLICATION IN TURBOMACHINERY AERODYNAMICS

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INTRODUCTION

Secondary flow is produced when a streamwise component of vorticity is developed from the deflection of an initially sheared flow. Such secondary flows occur when a developed pipe flow enters a bend, when a sheared flow passes over an airfoil of finite thickness or an airfoil of finite lift, or when a boundary layer meets an obstacle normal to the surface over which it is flowing (e.g. a wind blowing past a telegraph pole).

One of the most important engineering aspects of secondary flow occurs in axial turbomachinery aerodynamics, where boundary layers growing on the casing and hub walls of the machines are deflected by rows of blades, stationary and rotating. The authors' work in the field of secondary flow has been concerned largely with measuring and attempting to describe analytically these difficult flow problems in turbomachines, and this review concentrates on that area of the subject.

However, it is important for the fluid mechanicist to realize that the analysis of secondary flow has now reached the status of an important "classical" area of fluid mechanics comparable to potential flow theory. The development of the subject can be traced in a number of papers and reviews: Squire & Winter (1951), Hawthorne (1951, 1954, 1955, 1961, 1965, 1966, 1967), L. H. Smith (1955), Lighthill (1956), Marris (1963), Lakshminarayana & Horlock (1963), Hawthorne & Novak (1969).

We do not attempt here to provide a review of these reviews, but show instead how some of the major results may be applied in turbomachinery design and what limitations still exist in secondary-flow analysis.

We trace the development of secondary-flow theory and its application in the following way, thinking in terms of a sheared flow produced near the casing or hub wall of a turbomachine and deflected by a row of blades.

First, expressions are given for the streamwise vorticity generated along a streamline in a duct formed by the blade surfaces.

Then solutions for the secondary velocity fields are discussed. They have been classified by Hawthorne (1967). The two parameters of importance are the magnitude of the entry shear and the deflection of the flow. Thus four flows may be considered:

(i) *small shear, small disturbance* in which the Bernoulli surfaces are undistorted and the disturbance is irrotational;

(ii) *small shear, large disturbance* in which a primary irrotational flow convects the Bernoulli surfaces and the vortex filaments (this is referred to as the "secondary flow approximation" and is the one most commonly used to describe secondary flows in turbomachines);

(iii) *large shear, small disturbance* in which the disturbance is rotational and the Bernoulli surfaces are distorted (an example of this approximation is the shear flow past thin airfoils—this approach has not yet been widely used in turbomachinery aerodynamics and we briefly review it);

(iv) *large shear, large disturbance*, where very few solutions exist.

We discuss solutions of type (ii) in detail and type (iii) briefly.

Next we compare some of the analyses with carefully designed experiments on two-dimensional cascades, single three-dimensional twisted blade rows (stationary and rotating), and turbomachinery stages of two or three rows. The effects of various flow and blade parameters on secondary flow and losses are also discussed.

Finally, we discuss what can and cannot be done in predicting the secondary flows in multistage machines.

EQUATIONS FOR THE STREAMWISE VORTICITY

In this section we give general equations for the streamwise vorticity developed along *any* curved streamline, within a bent duct. We then derive simpler equations for a particular coordinate geometry.

Streamwise Vorticity—A General Statement

For an incompressible flow that has velocity $\mathbf{V}$ (scalar V) and vorticity $\boldsymbol{\omega}$ a purely kinematical relationship depending only on the continuity equation (div $\mathbf{V}=0$) has been given by Marris (1963) as a generalization of earlier work by Hawthorne (1951),

$$(\mathbf{V}\cdot\nabla)\left(\frac{\mathbf{V}\cdot\boldsymbol{\omega}}{V^2}\right) = V\frac{\partial}{\partial s}\left(\frac{\xi}{V}\right) \tag{1}$$

$$= \frac{2}{VR}[\mathbf{s}\times(\mathbf{V}\times\boldsymbol{\omega})\cdot\mathbf{n}] - \frac{\mathbf{s}}{V}\cdot\nabla\times(\mathbf{V}\times\boldsymbol{\omega})$$

Here $\mathbf{s}$ is the unit vector tangent to the streamline of local radius of curvature R and $\mathbf{n}$ is the unit vector along the principal normal to the streamline (i.e. in a direction away from the center of curvature, opposite to that of the centrifugal acceleration V^2/R), so that **s**, **n**, and **b** form a right-handed system (Figure 1a) where **b** is the unit vector along the binormal; and ξ and η are the components of vorticity in the directions s and n respectively. Since $-[\mathbf{s}\times(\mathbf{V}\times\omega)\cdot\mathbf{n}]/V$ is equal to the scalar component η of the vorticity along the principal normal, equation (1) may be written

$$\frac{\partial}{\partial s}\left(\frac{\xi}{V}\right) = -\frac{2\eta}{VR} - \frac{\mathbf{s}\cdot\nabla\times(\mathbf{V}\times\omega)}{V^2} \tag{2}$$

Only at this point do we need to introduce a momentum equation. In laminar flow of a fluid with kinematic viscosity ν, it is

$$\mathbf{V}\times\omega = \frac{\nabla P}{\rho} - \nu\nabla^2\mathbf{V} \tag{3}$$

where P is the total pressure and ρ the density. Equation (2) then becomes

$$\frac{\partial}{\partial s}\left(\frac{\xi}{V}\right) = -\frac{2\eta}{VR} + \frac{\nu\mathbf{s}\cdot\nabla^2\omega}{V^2} \tag{4a}$$

$$\simeq -\frac{2\eta}{VR} + \frac{\nu\nabla^2\xi}{V^2} \tag{4b}$$

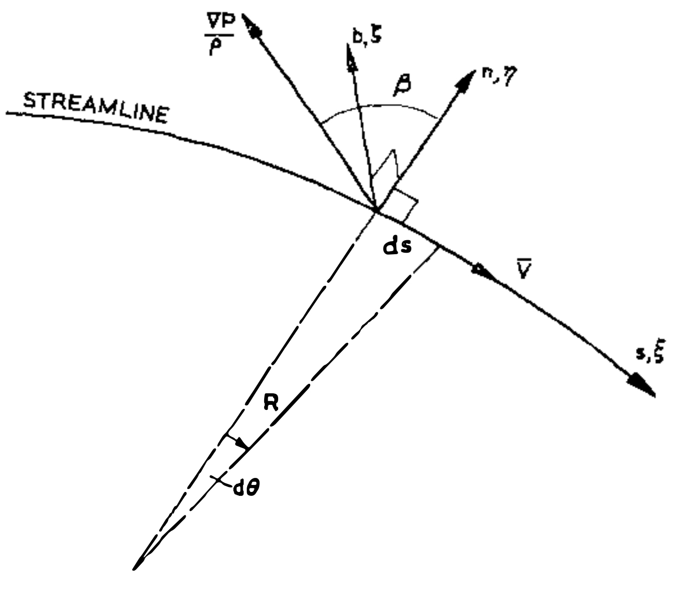

Figure 1a Development of streamwise vorticity—general coordinates.

A similar equation has been given by Hawthorne (1965). (If we can use an eddy viscosity ν_e in turbulent flow, then equation (4b) can be used with ν_e replacing ν.)

Equation (4a), given by Marris, is a general statement for the growth of secondary vorticity. An alternative form frequently used is obtained by substituting $V\eta = |\nabla P/\rho| \sin\beta$, where β is the angle between the perpendicular to the Bernoulli surfaces and the principal normal, into equation (4b), giving

$$\frac{\partial}{\partial s}\left(\frac{\xi}{V}\right) \simeq \frac{1}{V^2}\left[-2\left|\frac{\nabla P}{\rho}\right|\frac{\sin\beta}{R} + \nu\nabla^2\xi\right] \tag{5}$$

We have presented the streamwise vorticity equations for incompressible flow. More complex expressions for this vorticity, which are required in compressible flow, have been given by Hawthorne (1966) and Loos (1956) but are not reproduced here.

Simpler Derivations and Approximate Forms for the Streamwise Vorticity

Simpler statements of the secondary-vorticity equations are obtained from the more familiar equation for the vorticity in incompressible laminar flow,

$$(\mathbf{V}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{V} + \nu\nabla^2\boldsymbol{\omega} \tag{6}$$

obtained by taking the curl of equation (3) as a starting point.

The streamwise component of this equation is

$$V\frac{\partial\xi}{\partial s} + \frac{V\eta}{R} \simeq \xi\frac{\partial V}{\partial s} + \eta\frac{\partial V}{\partial n} + \zeta\frac{\partial V}{\partial b} + \nu\nabla^2\xi \tag{7}$$

where ζ is the vorticity in the b direction, and $\mathbf{s}\cdot\nabla^2\boldsymbol{\omega}$ has again been written as $\nabla^2\xi$. Since $\eta = \partial V/\partial b$ and $\zeta = -(\partial V/\partial n + V/R)$, it follows that

$$\frac{\partial}{\partial s}\left(\frac{\xi}{V}\right) \simeq -\frac{2\eta}{VR} + \frac{\nu\nabla^2\xi}{V^2}$$

which is Marris's equation, (4b).

Similarly, the equation for η in the n direction is

$$\frac{\partial}{\partial s}(V\eta) \simeq \frac{V\zeta}{\tau} + \nu(\nabla^2\eta) \tag{8}$$

in a flow of small deflection, where τ is the radius of torsion of the streamlines.

Squire and Winter's expression for secondary vorticity can be derived by assuming that $\nu = 0$, V is constant, and the radius of the streamline R is the same as that of the bend, so that $Rd\theta = ds$, where $d\theta$ is the elementary deflection of the streamline in the bend. Then equation (3) becomes

$$\frac{\partial\xi}{\partial\theta} = -2\eta \tag{9}$$

Louis (1956) has provided an interesting development of the basic equations to allow for the effect of viscosity. He considers a shear flow decaying owing to

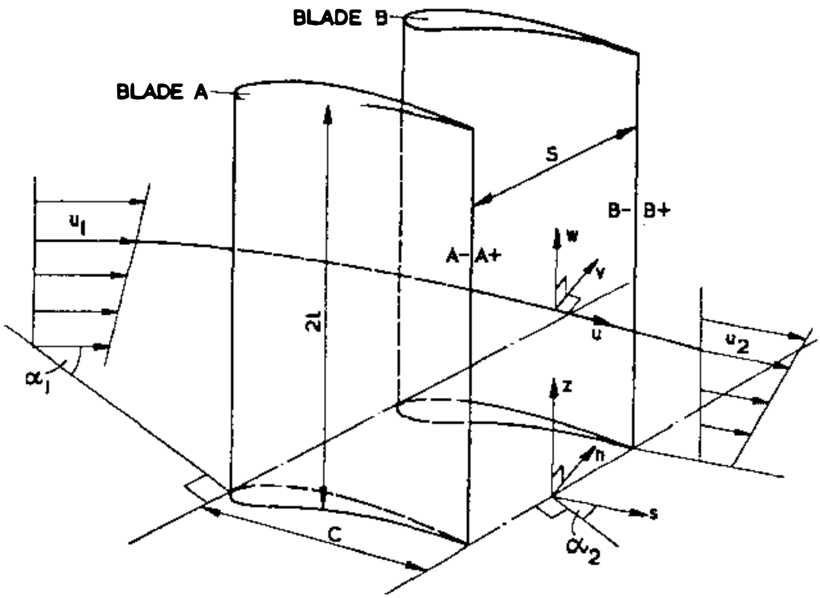

Figure 1b Cascade geometry.

viscous action as it is turned. He neglects the direct viscous effects on the streamwise vorticity in (7), and assumes that the effect of viscosity is to change η in (8). He eliminates the term on the right-hand side of (8) by assuming that the viscous action is the same as that in an undeflected two-dimensional flow (e.g. from an empirical law such as the decay of a turbulent wake, a two-dimensional solution, $V_{2D}(s, b)$, is known). If V_0 is the local velocity for inviscid potential flow through the bend, with unity velocity upstream, then Louis shows that

$$\frac{\xi}{V_0} = -2\int_0^s \frac{1}{V_0}\frac{\partial V_{2D}}{\partial b}\frac{ds}{R} + 2\int_0^s\left[\int_0^s V_{2D}\frac{\partial V_{2D}}{\partial b}\frac{\partial V_0}{\partial s}ds\right]\frac{ds}{RV_0^2} \quad (10)$$

Another approximate expression for the streamwise vorticity developed in a cascade (Figure 1b) has been given by Loos (1953) and is perhaps the most useful of all to the turbomachinery designer. In considering the deflection of a sheared flow of initial vorticity η_1 between a flow angle α_1 at inlet to a cascade and a flow angle α_2 at exit, Loos assumes the axial velocity to be unchanged, so that $V = V_1 \cos\alpha_1/\cos\alpha$ from continuity.[1] Using this relation in equation (4) he obtains

$$\xi_2 - \xi_1 = \frac{\eta_1}{\cos\alpha_1 \cos\alpha_2}\left[\alpha_2 - \alpha_1 + \frac{\sin 2\alpha_2 - \sin 2\alpha_1}{2}\right] \quad (11)$$

[1] α_2 is taken to be less than α_1, as in a compressor cascade. The flow is inviscid.

Again the small shear, large deflection approximation is implied, since it is assumed that the Bernoulli surfaces do not rotate, and the continuity equation is used within each plane.

Streamwise Vorticity in Rotating Channels

It is important for the turbomachinery designer to determine the streamwise vorticity in rotating as well as stationary channels. Marris (1966) has shown that equation (1) is valid in a rotating system if the unit vectors, velocities, and their curl are referred to the relative coordinate system. This is because the kinematic relationship must be the same, even though the momentum equation will be different. Thus equation (1) may be written, for incompressible flow in a channel rotating with angular velocity $\mathbf{\Omega}_b$ (see Marris 1966, as a generalization of earlier work by A. G. Smith 1957),

$$(\mathbf{W}\cdot\nabla)\left(\frac{\mathbf{W}\cdot\boldsymbol{\omega}'}{W^2}\right) = \frac{2}{WR'}[\mathbf{s}'\times(\mathbf{W}\times\boldsymbol{\omega}')\cdot\mathbf{n}'] - \frac{\mathbf{s}'}{W}\cdot\nabla\times(\mathbf{W}\times\boldsymbol{\omega}') \quad (12)$$

where $\mathbf{W}$ is the relative velocity, $\boldsymbol{\omega}'=\nabla\times\mathbf{W}$, and $\mathbf{s}'$, $\mathbf{n}'$ are unit vectors referred to the *relative* streamline of radius of curvature R'. For inviscid flow the equation of motion is

$$\frac{\nabla I}{\rho} = \mathbf{W}\times\boldsymbol{\omega} \quad (13)$$

where

$$I = p/\rho + \tfrac{1}{2}(W^2 - U_b^2)$$

and $\mathbf{U}_b = \mathbf{\Omega}_b\times\mathbf{r}$. Since $\mathbf{W}=\mathbf{V}-\mathbf{U}_b$ and $\nabla\times\mathbf{W}=\boldsymbol{\omega}'=\boldsymbol{\omega}-2\mathbf{\Omega}_b$ it follows that

$$\frac{\nabla I}{\rho} = \mathbf{W}\times(\boldsymbol{\omega}' + 2\mathbf{\Omega}_b) \quad (14)$$

If equation (14) is used in equation (12) (viscosity being ignored) then it may be shown that

$$\frac{\partial}{\partial s'}\left(\frac{\xi'}{W}\right) = -\frac{2}{W^2R'}\left|\frac{\nabla I}{\rho}\right|\sin\beta + \frac{2\mathbf{\Omega}_b\cdot\nabla W}{W^2} \quad (15)$$

where ξ' is the component of $\boldsymbol{\omega}'$ in the streamwise direction.

By writing $\boldsymbol{\omega}'=\boldsymbol{\omega}-2\mathbf{\Omega}_b$, we can express equation (15) in a form given by A. G. Smith (1957):

$$\frac{\partial}{\partial s'}\left(\frac{\xi_W}{W}\right) = -\frac{2}{W^2R'}\left|\frac{\nabla I}{\rho}\right|\sin\beta + \frac{2\Omega_b}{W^2}\left|\frac{\nabla I}{\rho}\right|\cos\delta \quad (16)$$

where ξ_W is the component of *absolute* vorticity $\boldsymbol{\omega}$ in the direction $\mathbf{s}'$ and δ is the angle between $\nabla I/\rho$ and $\mathbf{\Omega}_b$.

Again, these expressions for streamwise vorticity (equations 15 and 16 respectively) may be obtained more simply from the vorticity equation in rotating coordinates. The method is similar to that indicated in the earlier section (see Horlock & Lakshminarayana 1973).

For small rotation of the Bernoulli surfaces in a rotating cascade of blades, δ and β are approximately $\pi/2$ and only the first term is important. Then

$$\left(\frac{\xi_W}{W}\right)_2 - \left(\frac{\xi_W}{W}\right)_1 = -\int_0^{\theta'} 2\left|\frac{\nabla I}{\rho}\right| \frac{d\theta'}{W^2} \tag{17}$$

where θ' is the relative deflection. Thus if the *absolute* vorticity perpendicular to the *relative* streamline is η_{W_1}, and the velocity changes little in a small (relative) deflection, then

$$\xi_{W_2} - \xi_{W_1} = -2\eta_{W_1}\theta' \tag{18}$$

This is equivalent to the simple Squire and Winter statement for flow in stationary coordinates [equation (9)].

THE SECONDARY-FLOW APPROXIMATION (SMALL SHEAR, LARGE DISTURBANCE)

First we discuss solutions for the secondary flow based on the approximation of small shear and large deflection, and then some limitations of this approach.

Secondary Velocities and Angle Changes in a Cascade

The flow through a bend or a cascade of blades may be treated along the lines already described in the introduction. A uniform primary flow convects, stretches, and twists the vortex filaments. The secondary flow is regarded as a perturbation from the shear flow on undisturbed Bernoulli planes (i.e., transverse to the main primary flow direction) and the streamwise vorticity is obtained from the equations already given. Within the channel formed by the blades (Figure 1b) the secondary velocities (v, w) may be obtained from the secondary vorticity if it is assumed that $\partial v/\partial n$, $\partial w/\partial z \gg \partial u/\partial s$. (Note that we are now taking n and z as rectilinear coordinates at some location in the channel.) Then the continuity equation is

$$\frac{\partial v}{\partial n} + \frac{\partial w}{\partial z} = 0 \tag{19}$$

so that

$$-\xi = \frac{\partial v}{\partial z} - \frac{\partial w}{\partial n} = \nabla^2\psi \tag{20}$$

where $w = -\partial\psi/\partial n$, $v = \partial\psi/\partial z$, ψ being the secondary stream function. For a cascade channel (Figure 1b) at any section s, the cross section (width $S' \times$ length $2l$, where $S' = S \cos \alpha_2$) is rectangular and a solution to the equation has been given by Hawthorne (1955), for the Squire and Winter solution $\xi_2 = -2\theta_c du_1/dz$,

$$\psi = \sum_m \psi_m(z) \sin \frac{m\pi n}{S'} \tag{21}$$

where

$$\psi_m = -\frac{8\theta_c S'}{(m\pi)^2} \left\{ \frac{\sinh m\pi z/S'}{\sinh m\pi l/S'} \int_z^l \frac{du_1}{d\lambda} \sinh \frac{m\pi(l-\lambda)}{S'} d\lambda + \frac{\sinh m\pi(l-z)/S'}{\sinh m\pi l/S'} \int_0^z \frac{du_1}{d\lambda} \sinh \frac{m\pi\lambda}{S'} d\lambda \right\}$$

where θ_c is the turning angle of the cascade. This solution is valid for a plane cascade with symmetrical flow about mid span. It can be easily modified for other cases.

In particular the passage-averaged cross-flow velocity $\bar{v}_2$ is given by

$$\bar{v}_2 = \frac{1}{S'} \int_0^{S'} v dn = \frac{2}{\pi} \sum_{m=1,3,5\ldots} \frac{\psi_m'}{m} \tag{22}$$

and the cross-flow angle change (averaged across the pitch) is

$$\Delta\bar{\alpha}_2 = \frac{\bar{v}_2}{u_2} = \frac{\bar{v}_2}{u_1} \frac{\cos \alpha_2}{\cos \alpha_1} \tag{23}$$

Smith (1955) has called this type of solution the *channel theory* of secondary flow, since it is based on the determination of the streamwise vorticity within the channel. If the channel walls end abruptly then there is an interaction between the streamwise vorticity within the channel and the vortex sheets leaving the walls (or blades) that bound the channel. We consider the velocities at the trailing edge, and the vortex sheets thus created, in the next section.

Vortex Sheets Arising From Secondary Flow

To discover what happens at the trailing edge of a cascade blade, we may trace the path of the vortex filaments carried through the cascade. This has been done by several authors (e.g. Hawthorne & Armstrong 1955 and Smith 1955). Perhaps the clearest way of doing this is to use the drift function (t) introduced by Lighthill as in an earlier review by Hawthorne & Novak (1969).

The general drift function approach involves identifying the vortex filaments in the flow at some time far upstream before the flow is deflected. At time t later the vortex filaments are located again and lines of constant t may be drawn throughout the flow for the successive positions of the vortex filaments. The fluid particles are carried a distance ds in the direction of the velocity vector $\mathbf{V}$ in time dt. It follows that $\omega \cdot \nabla t = 0$ and $\mathbf{V} \cdot \nabla t = 1$.

For an inviscid, incompressible fluid

$$\frac{\nabla P}{\rho} \times \nabla t = (\mathbf{V} \times \omega) \times \nabla t = \omega = \nabla \times \mathbf{V} \tag{24}$$

and

$$\mathbf{V} = \nabla\phi - t\frac{\nabla P}{\rho} \tag{25}$$

which is a form of Clebsch's transformation, and ϕ is a potential function. Hawthorne & Novak (1969) apply the drift function to a weak linear shear flow $u_1(z)$ (with du_1/dz constant) past a long cascade, and use the secondary-flow approximation with the vortex filaments convected by the uniform primary flow. It may be shown that the components of V are $u=u_1\partial\phi_0/\partial s$, $v=u_1\partial\phi_0/\partial n$, $w=(\phi_0-t_0)\cdot(du_1/dz)$, where t_0 is the drift function of the plane primary flow, having potential ϕ_0. (Other additional velocities may be required to satisfy the Kutta condition precisely.)

Using these expressions, Hawthorne determines the strength of the trailing vortex sheet on blade B (Figure 1b) as

$$\begin{aligned}\Gamma_{wB} = w_{B+} - w_{B-} &= \frac{du_1}{dz}(\phi_{0_{B+}} - \phi_{0_{B-}} - (t_{0_{B+}} - t_{0_{B-}})) \\ &= \frac{d\Gamma}{dz} - \frac{du_1}{dz}\oint \frac{ds}{V_0}\end{aligned} \tag{26}$$

where Γ is the local circulation and the integral is taken around the airfoil surface in the direction of the circulation and is in general negative. Note that we take + as the suction surface, − as the pressure surface, of a compressor cascade (Figure 1b) and Γ_w is, in the same sense as ξ, positive in the flow direction.

The first term is the *shed circulation*, the gradient of the bound circulation along each airfoil, and the second is called the *trailing filament circulation*, caused by the stretching of the entering vortex filaments as they move over the surfaces of the airfoil.

Again, using the expression for w, Hawthorne shows that the total circulation enclosed within a section of unit height far downstream of the channel, where $v=0$, is

$$\Gamma_{\text{total}} = \frac{d\Gamma}{dz} + S'\frac{du_1}{dz}\left(\frac{\sin\alpha_2}{\cos\alpha_1} - \frac{\sin\alpha_1}{\cos\alpha_2}\right) \tag{27}$$

The averaged secondary circulation within the passage is therefore

$$\Gamma_{\text{total}} - \Gamma_w = \xi S' = \frac{du_1}{dz}\left[u_1\oint\frac{ds}{V} + S'\left(\frac{\sin\alpha_2}{\cos\alpha_1} - \frac{\sin\alpha_1}{\cos\alpha_2}\right)\right] \tag{28}$$

which is identical to a form given earlier by Hawthorne (1955). The channel secondary vorticity is

$$\xi = \frac{du_1/dz}{\cos \alpha_1 \cos \alpha_2}\left[\frac{u_1 \cos \alpha_1}{S}\oint \frac{ds}{V} + \frac{\sin 2\alpha_2 - \sin 2\alpha_1}{2}\right] \tag{29}$$

which is similar, but not identical, to the expression for secondary vorticity obtained by Loos, equation (11).

An alternative approach to this problem was given by L. H. Smith (1955), who traced the vortex filaments through in similar fashion. However, Smith *defines* the secondary vorticity as the difference between the actual component of the vorticity within the channel and that which would exist if the channel were of infinitesimal width, i.e. if the blades were closely spaced as in an actuator disc. (Smith's primary flow is the actuator disc case, whereas Hawthorne's primary flow is a uniform flow convecting the vortex filaments. However, if the shear is small then the differences in primary flow do not matter.)

Smith's analysis leads to an expression for his averaged secondary vorticity of the form

$$\xi S' = -\left[\frac{du_1}{dz} u_1 \frac{\Gamma_{VA}}{u_\infty^2} + \frac{d\Gamma_V}{dz}\right]\frac{dz_1}{dz_2} \tag{30}$$

where Γ_{VA} is the positive actual circulation (Smith's primary plus secondary circulation) around each vane or airfoil, Γ_V is the (Smith) primary circulation, dz_1 and dz_2 are the distances between the stream surfaces upstream and downstream, and u_∞ is the vector mean velocity through the cascade.

We may compare these expressions of Hawthorne [equation (28)] and Smith [equation (30)] for the secondary vorticity developed in a two-dimensional compressor cascade deflecting the flow from a flow angle α_1 to a flow angle α_2, in which the shear is weak, $\eta_1 = du_1/dz$, and $dz_1 = dz_2$. The terms $\oint ds/V$ and $-\Gamma_{VA}/u_\infty^2$ are clearly identical (both negative). Smith's term $d\Gamma_V/dz$ may be written

$$\frac{d\Gamma_V}{dz} = S\frac{d}{dz}(u_1 \sin \alpha_1 - u_2 \sin \alpha_2)$$

$$= S\left(\sin \alpha_1 \frac{du_1}{dz} - \sin \alpha_2 \frac{du_1}{dz}\right)$$

since α_1 and α_2 do not vary with z in the primary flow.

It is at this stage that the assumption about Smith's primary flow becomes critical. In that flow $P_1 \simeq P_2$, and with static pressures p_1 and p_2 constant, it follows that

$$\frac{du_2}{dz} \simeq \frac{\cos \alpha_2}{\cos \alpha_1}\frac{du_1}{dz}$$

Hence Smith's expression (30) for the secondary circulation becomes

$$\xi S' = u_1 \frac{du_1}{dz} \oint \frac{ds}{V} + S \frac{du_1}{dz} \left(\frac{\sin \alpha_2 \cos \alpha_2}{\cos \alpha_1} - \sin \alpha_1 \right) \tag{31}$$

and there is no inconsistency between L. H. Smith (31) and Hawthorne (28) for this simple case in which the primary flows of Smith and Hawthorne both have the vorticity normal to the streamline leaving the cascade. (Note that Smith's secondary vorticity as defined is not *always* along the streamline, whereas Hawthorne's is.)

If there is streamwise vorticity at entry to the cascade then the equivalence of the two approaches is not so immediately evident. Consider for example a "Beltrami" flow (one of uniform stagnation pressure but with streamwise vorticity, which makes the flow angle nonuniform) moving through a cascade of twisted flat plates that receive the skewed flow at zero incidence but do not deflect it at all. The general equation (4) shows that $\xi_2 = \xi_1$ where ξ_1 is the entry streamwise vorticity—the only component of vorticity. Thus, in Hawthorne's definition, the streamwise vorticity at exit is equal to that at entry. However, from Smith's equation (30), since $\Gamma_{VA} = \Gamma_V = 0$, the secondary vorticity is zero (using Smith's definition). However, primary (Smith) vorticity at exit is equal to that at entry, so that there is vorticity along the streamline at exit. Smith would call this streamwise vorticity primary vorticity, whereas Hawthorne would call it secondary vorticity.

For rotating coordinates L. H. Smith's statement is the same as that for stationary coordinates:

$$-(\xi_W - \xi_{W_{pr}}) = \left(W_1 \eta_{W_1} \frac{\Gamma_{VA}}{W_\infty^2} + \frac{d\Gamma_V}{dz} \right) \frac{dz_1}{dz_2} \tag{32}$$

where η_{W_1} is again the absolute vorticity resolved normal to the relative streamlines at entry and $(\xi_{W_{pr}})$ is the absolute vorticity along the relative streamline in the primary flow. Smith still defines his secondary vorticity as the difference between the actual absolute vorticity and the absolute vorticity of the primary flow, employing the Helmholtz laws to see how the absolute vorticity changes through a blade row, which may be rotating or stationary. A difficult point that arises is the question of what vorticity should be used to calculate the secondary velocities in the rotating channels. L. H. Smith again argues that the vorticity that should be used is the difference $\xi_W - \xi_{W_{pr}}$. His argument may be illustrated as follows:

If the (Smith) primary absolute flow velocity is $\mathbf{V}_{pr}$ and it is disturbed to produce a velocity $\mathbf{v}$ (absolute), then we may state the relative primary and total disturbed relative flows as

$$\mathbf{W}_{pr} = \mathbf{V}_{pr} - \mathbf{\Omega}_b \times \mathbf{r}$$

(33)

and

$$\mathbf{W} = \mathbf{V}_{pr} + \mathbf{v} - \mathbf{\Omega}_b \times \mathbf{r}$$

respectively. The disturbance flow in relative coordinates is therefore $\mathbf{W}-\mathbf{W}_{pr}=\mathbf{v}$, and its curl is $\boldsymbol{\omega}$, the vorticity of the absolute disturbance. To calculate the disturbance velocities at the trailing-edge plane of the rotor row, we must therefore resolve the absolute vorticity in the direction of the relative flow to obtain $(\xi_W-\xi_{W_{pr}})$.

We may again discuss the consequences of using the A. G. Smith type of equation (16), which gives the total vorticity generated along the relative streamline, and the L. H. Smith equation (32), which gives the secondary vorticity as an excess over primary vorticity, by reference to two simple examples. First, consider a uniform flow entering a rotating row of blades. Since $\nabla I/\rho=0$ at entry, it is clear from A. G. Smith's equation (16) that no absolute vorticity ξ_W can be developed even if the flow is deflected. For a (relative) deflection of this flow, L. H. Smith's equation (32) would give $-[\xi_W-\xi_{W_{pr}}]=d\Gamma_V/dz$, since $\eta_{W_1}=0$. But $d\Gamma_V/dz=\xi_{W_{pr}}$ in the primary flow so that ξ_W would be zero, as predicted by A. G. Smith. However, L. H. Smith would argue that in his own definition secondary vorticity does exist.

As a second example, consider a forced vortex flow entering a rotor, with tangential velocity everywhere equal to the blade speed so that the relative velocity is axial. The entry vorticity is now $2\Omega_b$ in the axial direction, and $\nabla I/\rho=\mathbf{W}\times 2\boldsymbol{\Omega}_b=0$ at entry. Suppose this flow moves through a set of rotating flat plates without causing relative deflection. Then η_{W_1} need not be zero, but since there is no deflection the A. G. Smith equation (16) gives the absolute secondary vorticity ξ_W as unchanged: $\xi_W=2\Omega_b$. L. H. Smith's equations give $(\xi_W-\xi_{W_{pr}})=0$, since Γ_{VA} and Γ_V are both zero. However, this is not inconsistent with A. G. Smith's result that $\xi_W=2\Omega_b$ since L. H. Smith's primary vorticity is nonzero: $\xi_{W_{pr}}=2\Omega_b$, and therefore $\xi_W=\xi_{W_{pr}}=2\Omega_b$. It appears therefore that the A. G. Smith and L. H. Smith equations are consistent, but that to determine the secondary flow in rotating channels the (L. H. Smith) primary streamwise vorticity must first be subtracted from the total secondary vorticity before the secondary velocities are calculated as a disturbance of the (L. H. Smith) primary flow.

Secondary Flow in Passages with Twisted Blades

In most turbomachinery the blades are twisted, giving rise to spanwise variations of the blade geometry, which are not allowed for in the cascade theories presented in the earlier sections. Ehrich (1955) developed a theory for predicting the secondary flow in a cascade of twisted blades. A more general approach, based partly on Hawthorne & Novak's (1969) analysis but written in s, n, z coordinates, is given here.

If the flow is collateral at entry, but the blade twist is represented by $\theta_e-\bar{\theta}_e$ where θ_e is the local turning angle, and $\bar{\theta}_e$ is the (*spanwise*) mean turning angle for the cascade, then the equation for the secondary stream function in the s, n plane perpendicular to the mean flow at exit is given by equation (20), but the boundary conditions are modified to: (i) $\psi(n, 0)=\psi(n, 2l)=0$ and (ii) $\partial\psi/\partial z(\pm S'/2, z)=-u_1(\theta_e-\bar{\theta}_e)$ since excess deflection $(\theta_e-\bar{\theta}_e)$ causes a cross flow against the n

direction. By writing $\psi=\psi^*(n, z)+\psi'(z)$ where $d\psi'/dz=-u_1(\theta_e-\bar{\theta}_e)$ we can recast equation (20) into

$$\nabla^2\psi^* = -\xi + \frac{d}{dz}(u_1(\theta_e - \bar{\theta}_e)) \tag{34}$$

If Squire and Winter's expression for ξ [equation (9)] is used, equation (34) reduces to

$$\nabla^2\psi^* = (3\theta_e - \bar{\theta}_e)\frac{du_1}{dz} + u_1\frac{d\theta_e}{dz} \tag{35}$$

This indicates that for small shear and small twist, if $\theta_e u_1^3=$constant the secondary flow would be eliminated completely. Thus, if it is desired to reduce secondary flow, θ_e should be increased when u_1 is decreased. A physical interpretation of this conclusion is that both the secondary flow and the nonuniform inlet velocity reduce the circulation in the low-velocity region and this can be compensated for by increasing the local turning. The effect is to make the circulation more uniform along the span, eliminating the trailing vorticity and the secondary vorticity.

Ehrich tested this hypothesis by conducting three sets of experiments in a cascade of space/chord ratio = 1.04 and aspect ratio = 2.08: (i) with no twist, but nonuniform inlet flow; (ii) with twist, but with uniform inlet flow; (iii) with both twist and nonuniformity such that $\theta_e u_1^3=$constant. His experimental results reproduced in Figure 2 clearly reveal a reduction in secondary flow by this twisting of the blades. The experimental distributions of stream functions (Figure 2) agree closely with those predicted from the solution of equation (35) [assuming that $\theta_e \simeq \bar{\theta}_e$ so that the first term on the right-hand side is $-2\theta_e(du_1/dz)$].

Ehrich used a moderate shear across the blade span, avoiding the danger of flow separation that usually exists near the end walls. Attempts to achieve this objective near the end wall increase the risk of flow separation and the consequent losses. It should be noted that Martin (1959) achieved an unseparated flow of almost uniform exit angle by reducing the blade outlet angle near the wall and letting the secondary flow produce the deflection there.

Hawthorne & Novak (1969) have carried out an analysis of the flow in an annular cascade with *axial* inflow (u_1), but with variation in outlet flow angle α_2 along the span, now taken as r. They resolve the secondary vorticity ξ_2 into tangential and axial components, the former ($\xi_2 \sin \alpha_2$) causing a radial gradient of axial velocity and the latter leading to an equation for a (Stokes) stream function describing the radial and tangential velocities in the trailing edge (r, θ) plane of the form

$$\nabla^2\psi = -\xi_2 \cos \alpha_2 \tag{36}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

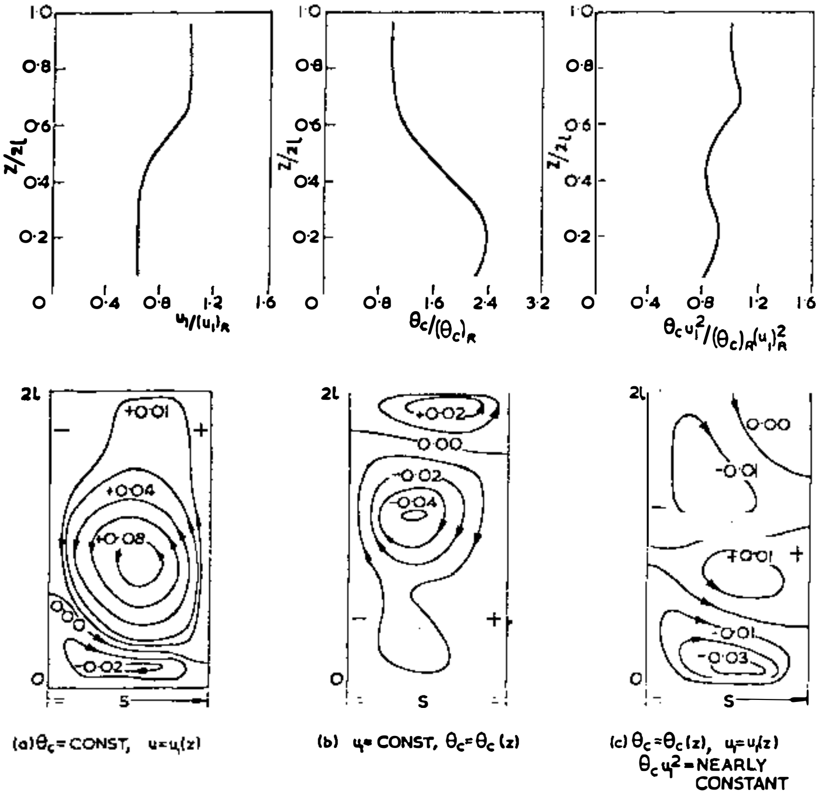

Figure 2 Measured secondary flow pattern with (a) nonuniform inlet flow, but no blade twist, (b) uniform inlet flow with blade twist, and (c) nonuniform inlet flow and blade twist. The numbers on the contour denote the values of $\psi/2(u_1)_R(\theta_c)_R l$ and subscript R denotes the reference value at inlet (Ehrich 1955).

Applying the Kutta-Joukowski condition, Hawthorne & Novak determine the tangential velocity required at each trailing edge from the axial velocity distribution. They subtract this tangential velocity, defining a new stream function giving zero tangential velocity at the boundaries of each blade channel. The equation for this modified stream function is

$$\nabla^2\psi^* = \frac{u_2 \cos \alpha_2}{r} \frac{d}{dr}(r \tan \alpha_2) - \xi_2 \sec \alpha_2 = F(r) \tag{37}$$

Later, Hawthorne (see Dixon 1972) reduced the above equation into Cartesian coordinates by writing

$$\frac{r}{r_{\text{hub}}} = \exp\left(\frac{2\pi z'}{N}\right), \qquad \theta = \frac{2\pi y'}{N}$$

where N is the number of blades. The solution is then given by

$$\psi^* = \sum_{m=1,3,5\cdots}^{\infty} \psi_m^* \sin(m\pi y') \tag{38}$$

where

$$\psi_m^*(z') = -\frac{4}{m^2\pi^2 \sinh m\pi l}\left\{\sinh m\pi z' \int_{z'}^{l} G(t) \sinh n\pi(l-t)dt + \sinh m\pi(l-z') \int_0^{z'} G(t) \sinh m\pi t dt\right\} \tag{39}$$

and

$$G(z') = F(r)\left(\frac{2\pi r}{N}\right)^2$$

This solution is similar to the earlier solution for a rectilinear cascade [equation (21)], but $G(z')$ now includes the effect of twist as well as secondary flow. These equations can be modified

Limitations and Developments of the Secondary-Flow Approximation

For a large turning cascade accepting a steep inlet velocity profile, the secondary-flow approximation loses validity. Distortion of the Bernoulli surfaces, spanwise flows, and effects of viscosity should be taken into account. Furthermore, the low momentum fluid is continuously transported towards the corner formed by the wall and suction surface, thus initiating wall stall in this region. We consider here modifications
these effects.

BERNOULLI-SURFACE ROTATION, DISPLACEMENT, AND VISCOUS EFFECTS If β is the angle between the direction of the principal normal and the normal to the Bernoulli surface, equation (5) becomes, for inviscid flow,

$$\frac{\partial}{\partial s}\left(\frac{\xi}{V}\right) = -\frac{2\eta_1 \sin\beta}{VR} \tag{41}$$

When the Bernoulli surfaces are not distorted as in the small-shear large-deflection case, β remains at its initial value of $\pi/2$. In practice, distortion does take place, and we may allow for it approximately in the secondary-vorticity equation although strictly the "secondary-flow approximation" does not permit us to do so.

Dean (1954) and Lakshminarayana & Horlock (1967a) have shown that, for moderate turning in which η_1 is unchanged through the cascade, this distortion is given approximately by

$$\beta \simeq \frac{\pi}{2} - \frac{\eta_1 \theta^2 c}{2\theta_c u} \tag{42}$$

where θ = local turning, θ_c = total turning, and c = chord length. Substitution of this into expression (41) yields

$$\left(\frac{\xi}{u}\right)_2 - \left(\frac{\xi}{u}\right)_1 = -2\int_0^{\theta_c} \frac{\eta_1}{u} \cos\left(\frac{\eta_1 \theta^2 c}{2\theta_c u}\right) d\theta \tag{43}$$

If u (θ, z) is known, viscous as well as Bernoulli surface rotation effects can be allowed for in calculating the strength of the streamwise vorticity at exit. Lakshminarayana & Horlock (1967a) have integrated the expression in equation (43). The effect of Bernoulli-surface rotation is to reduce the strength of the secondary vorticity, its effect becoming appreciable for large values of $(\eta_1 c/u_1)$ and θ_c.

At small outlet angles the secondary flow in the blade passage largely determines the variation in outlet angle downstream. At large outlet angles, the secondary flow effects are masked by large spanwise flows associated with changes in axial velocity profile referred to as *displacement effects*. Along a streamline outside the blades, the flow angle changes with the axial velocity, the tangential velocity being conserved. Hawthorne & Armstrong (1956) developed an analysis for predicting these effects using actuator-disc theory. Their expression for the effects of secondary flow (without Bernoulli-surface rotation) and spanwise displacement is given by

$$\begin{aligned}\overline{\Delta\alpha_2} = {} & \frac{2(m+1)\cos\alpha_2}{\pi U_1 \cos\alpha_1} \sum_{m=1,3\cdots}^{\infty} \frac{\psi_m'}{m} \\ & - \frac{\sin 2\alpha_2}{4}\left(\frac{\cos^2\alpha_2}{\cos^2\alpha_1} - 1\right)\left(1 - \left(\frac{U_1}{u_1}\right)^{p+1}\right)\end{aligned} \tag{44}$$

where $p = \tan^2\alpha_2/(2+\tan^2\alpha_2)$ and U_1, α_2 are the entry velocity and exit angle outside the shear region. Here ψ_m' is the derivative of a secondary stream function given by equation (21).

The authors (1967a) attempted to incorporate all these effects in the prediction of outlet angles and velocities in a cascade with an artificially
stream (Lakshminarayana & Horlock 1967b). The viscous effects were accounted for in an approximate way by taking into account the diffusion of the wake up to the actuator disc. The difference between this approach and that of Louis (1956)

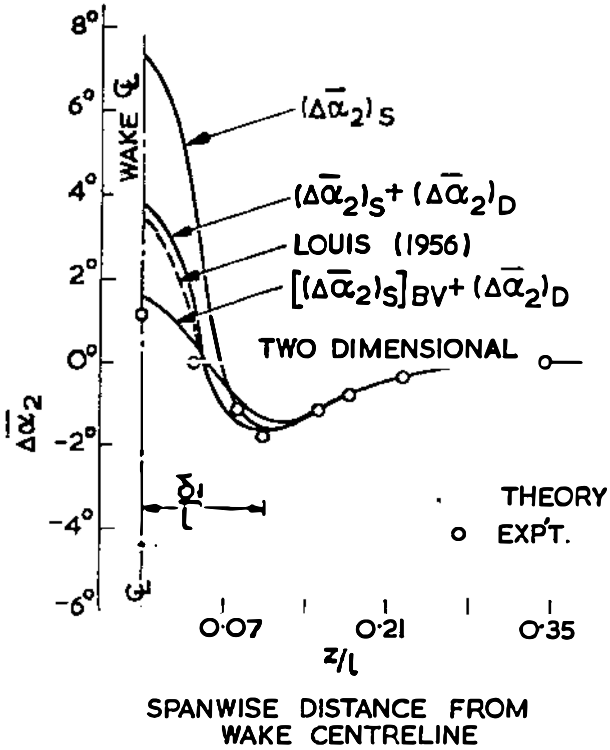

Figure 3 Comparison between experimental and predicted changes in average outlet in a cascade (Lakshminarayana & Horlock 1967a).

is that the latter allowed for continuous change in shear and calculated the streamwise vorticity at several chordwise stations. The authors' approach was first to calculate the velocity profile at the actuator plane, and then to estimate the secondary vorticity from equation (43) using the value of η_1 derived from this profile.

The authors' (1967a) results are shown plotted in Figure 3. The contribution due to (i) secondary flow with no rotation of Bernoulli surfaces, $(\Delta\alpha_2)_S$, (ii) secondary flow and displacement effect, $(\Delta\alpha_2)_S+(\Delta\alpha_2)_D$, (iii) secondary flow (with Bernoulli-surface rotation and viscous effects) and displacement effects, $(\Delta\alpha_2)_{BV}+(\Delta\alpha_2)_D$, are shown separately. A calculation using Louis' equation (10) is also shown. It is clear that these effects substantially reduce the overturning. The final calculations show good agreement with experimental results, but the interaction between these various effects is such that their linear superposition may not be justified. In particular, the large displacement effect may not be representative of boundary-layer flows.

SECONDARY FLOW AND THREE-DIMENSIONAL BOUNDARY-LAYER THEORY The secondary-flow
viscosity allowed for by Marris in equations (4) and the modifications suggested in the previous sections. The theory of three-dimensional boundary layers has been reviewed elsewhere, notably by Joubert (1967) in the book edited by Sovran. We may also note Emmons' comment on that paper and the parallel paper by Hawthorne (1967) on secondary-flow analysis: "I somehow have the feeling they were both talking about the same thing, at least in some regions, although it is quite obscure at the moment."

Regions of low viscosity One of the striking results of work on three-dimensional boundary layers is the so-called triangular polar plot of Johnston (1960). He found experimentally that in the outer part of the boundary layer the cross flow is given by

$$\frac{v}{U} = -A\left(1 - \frac{u}{U}\right) \tag{45}$$

and he explained this analytically by what was essentially inviscid secondary-flow analysis, showing that $A = 2\theta$. Johnston's result follows quickly from the Squire and Winter result for secondary vorticity

$$\xi = -2\theta\frac{du}{dz} = \frac{\partial w}{\partial n} - \frac{\partial v}{\partial z} \tag{46}$$

In an unbounded three-dimensional boundary layer (or one where the walls are far apart), $\partial v/\partial z \gg \partial w/\partial n$ and $2\theta du/dz = dv/dz$. Integration of this equation with the boundary condition that $v=0$ where $u=U$ yields $v=-2\theta(U-u)$, which is Johnston's result.

This similarity between the results of inviscid secondary-flow analysis and the cross flow in the outer part of a boundary layer has been extended by Horlock (1971); he has solved approximately the secondary-flow equation (20), taking $\xi=-2\theta du/dz$ and assuming the simple form for the secondary velocities v and w given by Mellor & Wood (1971), w varying linearly across the pitch and v parabolically. He takes the average of equation (20) across the pitch to obtain

$$\bar{v} = -2\theta(U-u) + k\delta^{*}\exp(-kz) \tag{47}$$

where $k=\sqrt{12}/S'$ and δ^* is the streamwise displacement thickness of the boundary layer.

This expression shows that as the distance between the walls $S'\rightarrow\infty$, Johnston's result is obtained. The mean angle variation across the pitch given by this ap-

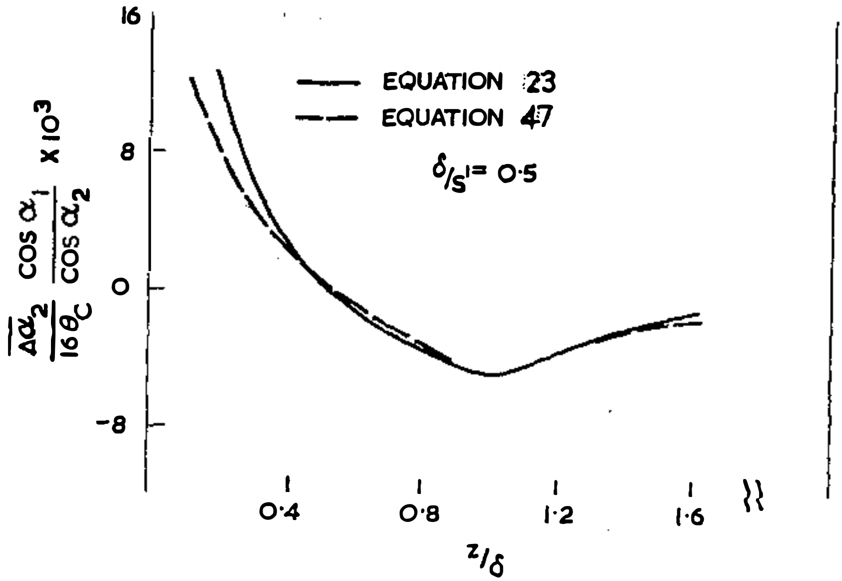

Figure 4 Comparison between Hawthorne's channel theory and Horlock's approximate solution.

proximate analysis is shown in comparison with Hawthorne's exact solution in Figure 4 for a cascade of large span. The advantage of Horlock's result lies in the relatively simple form of equation (47) and this is being used in three-dimensional integral boundary-layer calculations.

Boundary layer transport Perhaps the most important viscous effect on cascade flows is the sweeping of the wall boundary layers toward the suction surface of the blade. This results in concentration of low-momentum fluid in the corner between that surface and the wall, often leading to local separation or wall stall. Ehrich & Detra (1954) first made an estimate of this effect, calculating the local movement of fluid particles from inviscid secondary-flow analysis (the secondary-flow approximation). Horlock et al (1966), Dean (1954), and Lakshminarayana & Horlock (1967a) have estimated simply the rotation of the Bernoulli surfaces, but their analysis is not sufficient to deal with this problem. Horlock et al (1966) estimated the secondary rotation in the trailing-edge plane with the corner arbitrarily blocked off in a wall-stall region; Dring (1971) has recently calculated the boundary-layer development in the channel and the amount of fluid swept into the corner.

None of these approaches can satisfactorily explain the onset of wall stall. However, it may well be that this phenomenon is seen more often in cascades than in compressors, where streamwise vorticity at entry may oppose the generation of secondary vorticity and reduce the rotation of the Bernoulli surfaces and the sweeping of the boundary layer into the corner.

THE AIRFOIL TYPE OF ANALYSIS (LARGE SHEAR, SMALL DISTURBANCE)

As explained in the introduction, the airfoil type of analysis is based on the concept of a primary flow $u_1(z)$ (which is the unperturbed velocity in the absence of blades) that passes over a series of lifting lines or surfaces. This theory falls into the category of "large shear—small disturbance" approximations described by Hawthorne (1967) and is essentially based on linearization of the equations of motion. If the disturbance velocities are small, conventional thin-airfoil theory approximations are made in addition to the assumption that the flow is inviscid and isentropic. The analyses for a cascade (Honda 1961, Namba 1969) are similar to that proposed by von Kármán & Tsien (1945) for a single airfoil (lifting line) in a shear flow, but the cascade case is mathematically very complex.

The lifting-line approach is based on the use of a pressure dipole with its axis normal to the undisturbed stream, whereas in the lifting-surface theory a sheet of pressure dipoles is used to replace the cascade of blades (Figure 5). The basic equations of Namba (1969) will be given, and except for some minor changes his notation will be used throughout. While Namba (1969) distributes singularities along the direction of the upstream velocity as shown in Figure 5, Honda (1961) distributes them along the direction of the vector mean velocity (the vector mean of inlet and outlet velocity). The differences in the expressions of Honda and Namba are mainly due to this effect. In any case, the difference in the two approaches is a second-order effect, since the small-perturbation assumption means very low turning. The chord is taken as unity, so that the space-chord ratio is S and the aspect ratio is $2l$, i being the angle of incidence.

If u, v, w are now the *perturbation* velocities [from the inlet value $u_1(z)$] and p the perturbation in static pressure over the mean of inlet and exit pressure, linearization of the equations of motion and energy leads to the following equation:

$$[1 - \{M_0(z)\}^2] \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} - \frac{2}{M_0(z)} \frac{dM_0(z)}{dz} \frac{\partial p}{dz} = 0 \qquad (48)$$

where $M_0(z)$ is the value of the (unperturbed) Mach number far upstream. The boundary conditions to be satisfied are

(i) $\dfrac{\partial p}{\partial z} = 0 \qquad \text{at } z = 0, 2l$

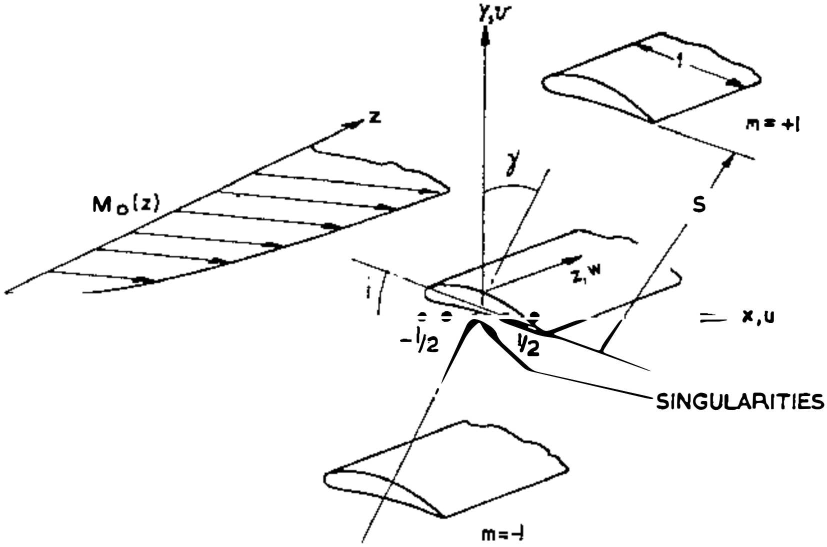

Figure 5 Notation for airfoil theory.

(ii) $p = \mp \dfrac{\Delta p}{2}$ as $x \to \mp\infty$

where Δp is the pressure rise across the cascade

(iii) the kinematic condition on the blade surface (camber),

$$\left(\frac{v}{u_1}\right)_{y=mS\cos\gamma} = f(x - mS\sin\gamma) \quad \text{for } |x - mS\sin\gamma| < \tfrac{1}{2} \text{ and } m = 0, \pm 1, \pm 2 \cdots \tag{49}$$

where γ is the stagger angle of the cascade (Figure 5) and $f(x)$ is the slope of the mean camber surface.

Using the method of separation of variables, we can express the solution of equation (49) in the following integral form:

$$p(x) = -\sum_{m=-\infty}^{m=+\infty} \operatorname{sgn}(y - mS\cos\gamma) \int_{-1/2}^{+1/2} d\hat{x} \int_0^\infty \Big[\cos\{k(x - \hat{x} - mS\sin\gamma)\} \sum_{n=0}^{\infty} \exp(-\lambda_n(k)\,|\,y - mS\cos\gamma\,|)\, F_n(\hat{x}, k)\, Y_n(z, k)\Big] dk \tag{50}$$

where $Y_n(z, k)$ and $\lambda_n(k)$ are eigenfunctions and eigenvalues respectively of the following boundary-value problem:

$$\frac{d}{dz}\left(\frac{1}{M_0{}^2}\frac{dY}{dz}\right)+\frac{1}{M_0{}^2}(\lambda^2-(1-M_0{}^2)k^2)Y=0 \tag{51}$$

with

$$\frac{dY}{dz}=0 \quad \text{at} \quad z=0,\ 2l$$

and

$$F_n(x, k)=\frac{1}{2l}\int_0^{2l}\frac{Y_n(z, k)}{M_0{}^2}\Delta p_s(x, z)dz$$

where Δp_s is the pressure jump across the lifting surface, $p(x)$ is the pressure perturbation at any point x, and $\hat{x}$ is the location of the pressure dipole.

In equation (50), $m=0$ corresponds to the contribution of the zeroth blade, and the remaining terms are due to interference effects of the adjoining blades. Methods of finding the coefficients F_n by using a Glauert type of trigonometric series are given by Namba (1969). Expressions for the local lift and lift coefficient are given by

$$L(z)=2\pi\sum_{n=0}^{\infty}\bar{F}_n(0)Y_n(z, 0) \tag{52}$$

$$C_L(z)=\frac{2L(z)}{\rho[u_1(z)]^2} \tag{53}$$

where

$$\bar{F}_n(0)=\int_{-1/2}^{+1/2}F_n(x, 0)dx$$

INCOMPRESSIBLE LIFTING-SURFACE SOLUTION The solution for the incompressible flow through a lifting surface can be obtained from this analysis simply by replacing M_0^2 with (u_1^2/a^2) and taking the limit as $a^2\to\infty$, with $u_1(z)$ fixed. The equations of Namba (1969) then reduce to those of Honda (1961), whose analysis is for incompressible flow. (See Namba 1969 and Nally & Hawthorne 1969.)

LIFTING-LINE SOLUTION In lifting-line theory only one pressure dipole is used at $x=0$. The governing equation and the boundary conditions are the same, except that the kinematic condition [equation (49)] is now replaced by

$$\lim_{\epsilon\to 0}\left\{[p]_{y=mS\cos\gamma-\epsilon}-[p]_{y=mS\cos\gamma+\epsilon}\right\}=\delta(x-mS\sin\gamma)L(z) \tag{54}$$

where δ is the Dirac delta function. The solution of this problem [similar to equation (50), without the integration over $\hat{x}$ and the dependency of F_n on $\hat{x}$] is given by Namba & Asanuma (1967).

Numerical Results from Airfoil Theory

Namba (1969) has investigated theoretically the effect of shear on a cascade with $S/c=1.0$, $\gamma=60°$, and aspect ratio $=2.5$ at various inlet Mach numbers for the incident Mach number profile $M_0(z)=(M_0)_{z=0}\exp(z/2l)$. His calculations are reproduced in Figure 6 and compared with the results of lifting-line theory (Namba & Asanuma 1967). In these figures the lift coefficient is normalized with respect to the corresponding value in two-dimensional incompressible flow, $C_{L_{2D}}$. The agreement between the lifting-surface and lifting-line theory is good at very low Mach numbers but it becomes progressively worse as the sonic point is reached ($z=2l$ in Figure 6).

Some of the other conclusions derived by Honda (1961), Namba & Asanuma (1967), and Nally & Hawthorne (1969) on the basis of airfoil theory are:

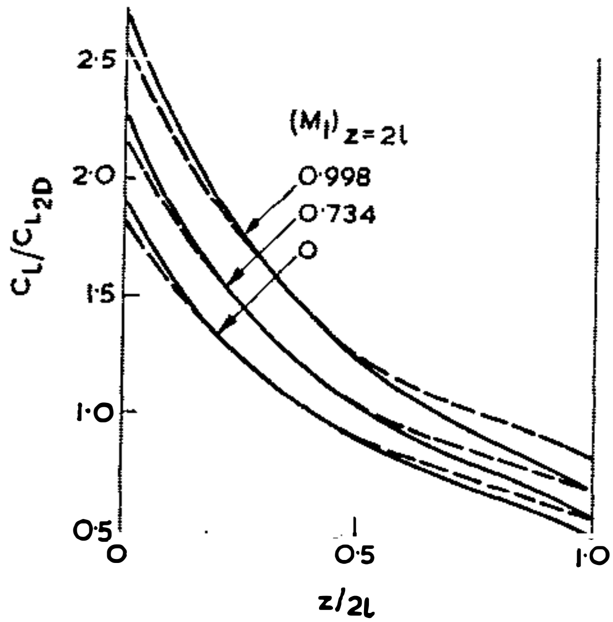

Figure 6 Spanwise distribution of the local lift coefficient in shear flows: lifting-surface theory; - - - - lifting-line theory. A cascade of flat plates of $S=1.0$, $\gamma=60°$, and $2l=2.5$ (Namba 1969).

(1) For the same shear (velocity gradient du_1/dz) and δ (boundary-layer thickness or extent of shear in the spanwise direction), uniformity in the distribution of local lift coefficient C_L is maintained as the spacing is decreased for constant chord. (However, note that δ/S is changed in such a calculation and this is probably the dominant parameter rather than space-chord ratio.)

(2) The kinetic energy in secondary flow decreases almost linearly with decrease in blade spacing, all other parameters being held constant.

(3) Increase in cascade blade stagger increases not only the spanwise nonuniformity in the C_L distribution but also the kinetic energy in the secondary flow.

(4) For the same S/c, entry shear, and δ, the effect of increasing the aspect ratio (by decreasing the chord length, keeping the blade length the same) is to weaken the secondary flow, thus bringing about a more uniform distribution of the spanwise lift coefficient and deflection angles.

(5) The sharp increase in lift coefficient near the low-velocity region (Figure 6) cannot be reduced by reducing the angle of attack to zero in that region. A considerable amount of twist, so as to reverse the sign of the angle of attack in this region, is required to overcome the large local increase in C_L.

(6) The spanwise distribution of the lift coefficient is weakly dependent on Mach number but shows a strong dependence on upstream velocity and entropy gradients.

Comparison with the Secondary-Flow Approximation

Namba (1967 and private communication) has shown that the circulation (Γ_{wl}) in the wake downstream derived from lifting-line theory and the circulation (Γ_{ws}) derived from the secondary-flow approximation are related by

$$\Gamma_{w_s} - \Gamma_{w_l} = O\left[\left(\frac{du}{dz}\right)^2\right] + O[\theta^2]$$

According to the assumptions made in airfoil theory $O(\theta^2)$ is small, and the secondary-flow approximation is based on small shear gradients, so that $O[(du/dz)^2]$ is small. Thus, within the limitations of the theories, both approaches lead to the same circulation in the wake far downstream.

EXPERIMENTAL RESULTS

There exists a large amount of experimental data relating to secondary flow. Some experiments were specifically aimed at checking the theory; several other results reveal the physical phenomena and the effect of cascade and flow parameters on flow losses.

Comparison with Theory: Cascades

Several investigators have carried out carefully designed experiments with inlet shear flows not only to check the theory but also to understand the basic phenomena. The inlet shear, generated by artificial means, was carefully selected to avoid flow separation, which is beyond the scope of present theories.

Hawthorne & Armstrong (1955) carried out their experiments in a cascade of turbine impulse blades of space-chord ratio unity, aspect ratio $2l/c=3$, $\alpha_1=35°$, and $\alpha_2=-37°$. The ratio of lift coefficients $C_L/C_{L_{2D}}$ (where C_L is the local lift coefficient and $C_{L_{2D}}$ is the corresponding two-dimensional value, both normalized with respect to local inlet dynamic head) is plotted in Figure 7 and compared with the predictions of Honda's (1961) lifting-surface theory and Hawthorne's small-shear, large-deflection analysis. The latter agrees better with the experiment, the agreement with airfoil theory being only qualitative. This is not surprising, since the airfoil theory is limited to very low turning. These results clearly reveal the limitations of the airfoil theory in practical configurations.

Lakshminarayana & Horlock (1967b) carried out their experiments in a cascade with space-chord ratio unity, $2l/c=4.83$, $\alpha_1=52°$, and $\alpha_2=31°$. The shear at

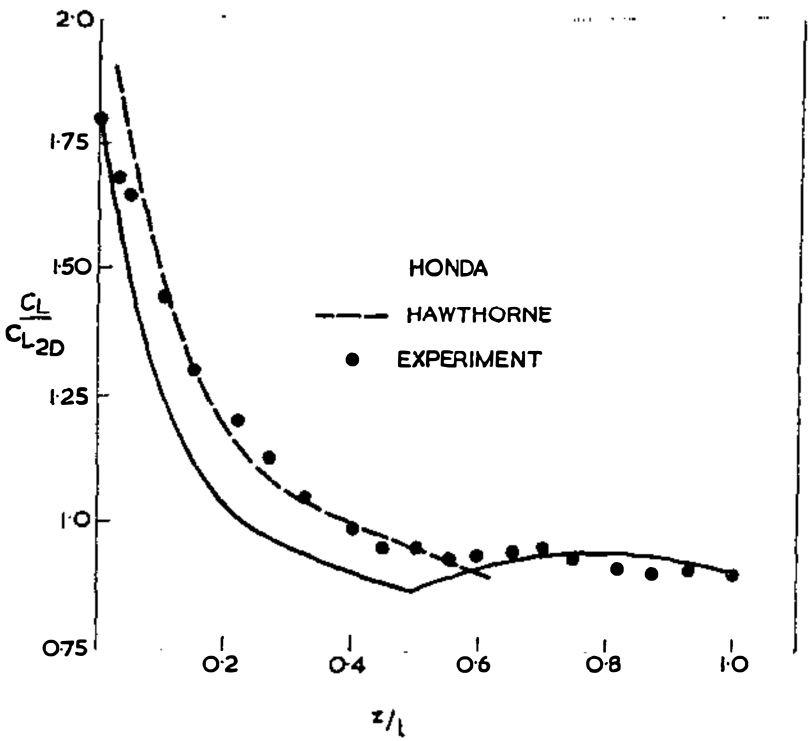

Figure 7 Comparison between theory (Honda and Hawthorne) and experiment (Hawthorne and Armstrong).

the inlet was obtained by placing a perforated metal plate upstream of the cascade. In Figure 3 the authors' measurement of the change in outlet angle ($\overline{\Delta\alpha_2}$) is compared with (i) Louis' (1956) theory and (ii) the authors' (1967a) calculation (which includes the effect of Bernoulli-surface rotation as well as an approximate estimate of viscous effects). The agreement with the authors' prediction is quite good. The difference between Louis' prediction and experiment is due to the neglect of Bernoulli-surface rotation and the displacement effect, neither of which is allowed for in his theory.

Namba & Asanuma (1967) carried out experiments in a cascade with high subsonic inlet Mach numbers to check their lifting-line analysis. The cascade parameters were space-chord ratio unity, $2l/c=1.25$, incidence $i=-4°$, stagger angle $\gamma=40°$, and blade camber $=16°$. Blade pressure distributions were measured for three inlet Mach number profiles $M_o=a\exp(1.45z/2l)$, with $a=0.075$, 0.15, and 0.23. The agreement between measured and predicted lift coefficient (Figure 8) is good at low subsonic Mach numbers, but the differences are appre-

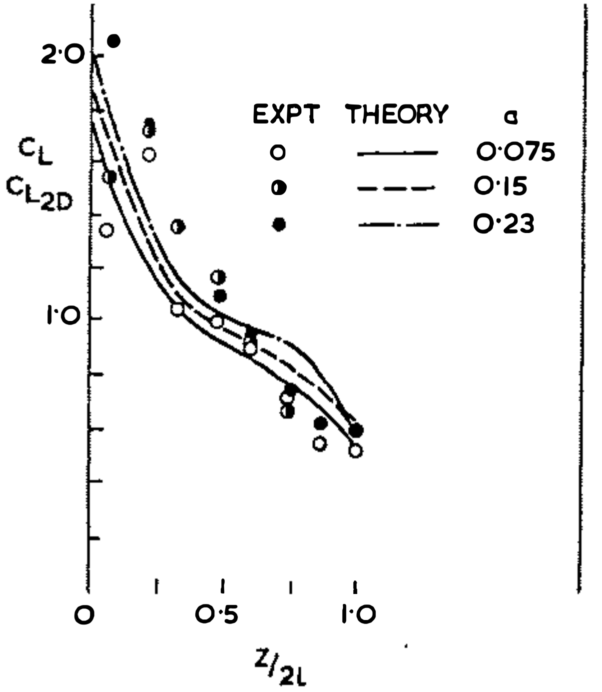

Figure 8 Spanwise distribution of local lift coefficient for a cascade ($\gamma=40°$, $S/c=1.4$, $2l=1.25$, $i=-4°$) (Namba & Asanuma 1967).

ciable at high subsonic Mach numbers. The discrepancy between theory and experiment near the wall (low values of $z/2l$) may be due to wall stall. Since the secondary flow is from pressure to suction surface in this region, low-energy fluid tends to accumulate and initiate the wall stall. In general, the agreement between theory and experiment is good, which establishes the validity of the analysis for low-turning cascades.

Comparison with Theory: Guide Vanes, Rotor and Stator

A number of experiments have been carried out to measure the extent of the secondary flow caused by the annulus wall boundary layer in one, two, or three blade rows of a turbomachine. Some comparisons have been made between these measurements and analytical predictions of the flow.

In an early experiment, Horlock (1963) tested a *stationary* row of twisted guide vanes with axial flow at entry, measuring the radial distribution of the mean outlet angle $\bar{\alpha}_2$. He made a two-dimensional secondary-flow calculation based on the tip-section blade geometry and the measured entry velocity, correcting the (three-dimensional) design flow angles by these calculations and obtaining fair agreement, as there was no wall separation in the row. The calculation would now best be done by the Hawthorne & Novak (1969) method for twisted rows, using the transformation solution given by equation (38). (See Dixon 1972 and Gregory-Smith 1970.)

For an isolated *rotating* row, Gregory-Smith (1970) has similarly measured the inlet and outlet boundary layers and the outlet angle distribution. He used (i) the (measured) *outlet* vorticity, η_{W_2}, (ii) a secondary vorticity, based on η_{W_2} and equation (16) for calculating ξ_{W_2}, and (iii) the approach suggested by Hawthorne & Novak (1969) for twisted blades, but using a finite-difference method to solve for the secondary velocities in the r, θ plane of the trailing edges. Gregory-Smith thus obtained the apparently perfect prediction of outlet angle distribution shown in Figure 9 in comparison with his experimental results. However, the use of η_{W_2} is questionable, as is the neglect of tip-clearance effects and Bernoulli-surface rotation. Lakshminarayana (1970) allowed for tip-clearance effects as well as for secondary flow in predicting the outlet angle measured at the exit of a General Electric rotor, and obtained qualitative agreement. He used the mean of the measured inlet and outlet velocity profiles in deriving the values of ξ_{W_2}.

The importance of combining a secondary-flow analysis with a boundary-layer approach to give η_{W_2} is emphasized here; a designer would not be able to *measure* η_{W_2} but would have to calculate it, if he were to take full account of the influence of secondary flow in his design. An attempt to calculate the boundary-layer development through this rotor (see Horlock & Hoadley 1970) is illustrated in the same figure. The cross flow was assumed to be of the Prandtl–Mager type, and although the boundary-layer thickness is closely estimated, the cross flow (or the outlet angle) is not so well predicted by this boundary-layer analysis as by the secondary-flow approach of Gregory-Smith, with an assumed velocity gradient $\eta_{W_2} = du_2/dr$. What is next required is the introduction of the angle variation

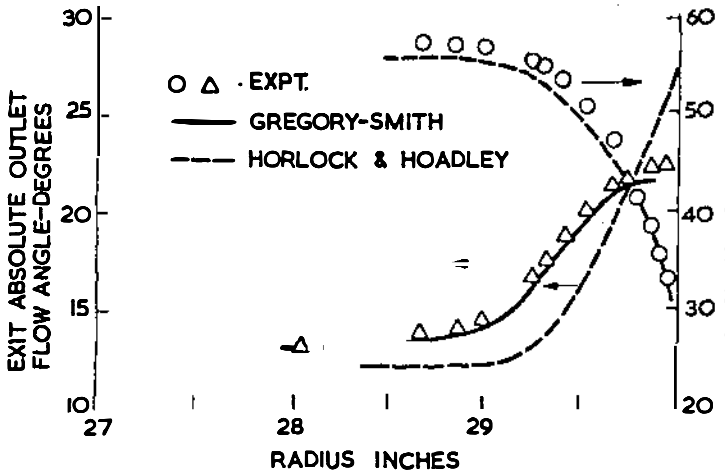

Figure 9 Axial velocity and outlet angles near the annulus wall for an isolated rotor (flow coefficient 0.7).

given by secondary-flow analysis into a boundary-layer prediction method, and this is being attempted.

Finally, if detailed secondary-flow calculations are to be successful in multistage compressors, account must be taken of the secondary vorticity produced by one row at entry to the next. An early attempt was made to do this by Horlock (1963), who again based his calculations on a two-dimensional solution for the secondary-stream function using the geometry of the tip sections of the three-stage compressor in which the flow was measured. The streamwise and normal vorticity components at the exit from the guide vane row were resolved into the relative flow direction to obtain ξ_{W_1}, before a simplified form of A. G. Smith's equation [in the form of equation (16)] was applied to the rotor row to obtain ξ_{W_2}. Figure 10 shows the angle variation calculated at rotor exit in this way in comparison with experimental measurements. The important thing to note here is that the large streamwise (relative) secondary vorticity at rotor *entry* leads to *underturning* at the tip of the rotor. The generation of vorticity within the row fails to reverse the direction of the entering streamwise vorticity, which comes from the preceding row.

L. H. Smith's equation (30) may be used in a similar fashion to trace the development of secondary vorticity (presumably the flow would be averaged downstream of one row to obtain axially symmetric velocities and Smith's primary vorticity at entry to the next). Smith (see Horlock 1963) has emphasized the limitations of Horlock's approach—that of assuming that the entry vorticity to

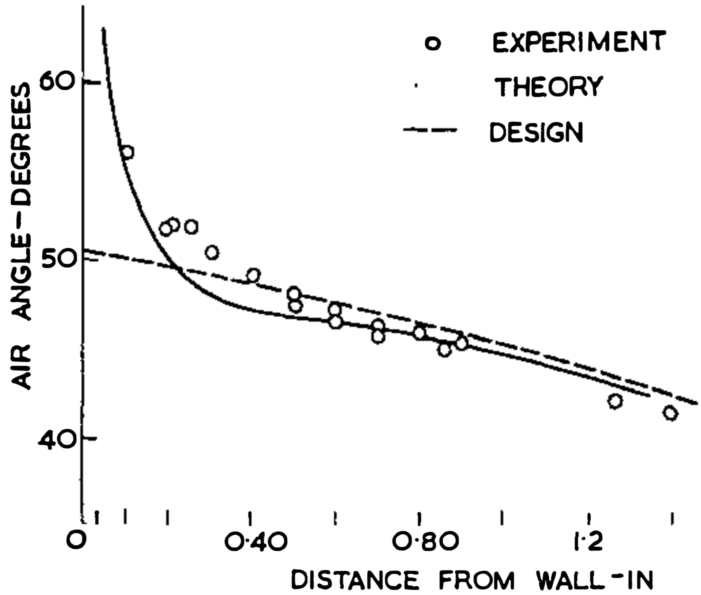

Figure 10 Measured and predicted outlet angles (relative) at the exit of a rotor (with guide vanes) (Horlock 1963).

the next row is simply the resolved part of the channel vorticity from the previous row.

Effect of Flow and Blade Geometry on Secondary Flow and Losses

Lakshminarayana & Horlock (1963), Horlock (1966), Balje (1968), and Dunham (1970) made attempts to examine the large amount of experimental data available from both compressor and turbine cascade tests, with a view to offering physical understanding of the phenomena and to producing a loss correlation. Dunham (1970) has made a comprehensive attempt to clear up some of the chaos that exists on the effect of blade and flow parameters on losses in cascades. No attempt will be made here to review the entire spectrum of data; we highlight some of the earlier conclusions and offer, whenever possible, some new interpretations of the earlier as well as the new data.

Secondary flow and losses may be expected to be functions of U_1, δ_1, H_1, c, l, S, ρ_1, μ, a_1, ξ_1, and flow geometry. These independent parameters can be grouped into nondimensional forms such as ($\rho_1 U_1 c/\mu$, U_1/a_1, δ_1/S, H_1, l/c, S/c, $\xi_1\delta_1/U_1$, flow geometry), where the first two groups are Reynolds number (R_N) and Mach number (M) respectively, and H_1 is the entry shape factor which indirectly takes into account du_1/dz. Note that $S'/c=(S \cos \alpha_2)/c$ has not been included as an independent variable but is absorbed within the parameters S/c and "flow geometry." The local secondary outlet-angle change $\overline{\Delta\alpha_2}$ and loss coefficient $\bar{\zeta}$, passage averaged at a distance z from the wall, can therefore be written as

$$\overline{\Delta\alpha_2},\ \bar{\zeta}(\text{or } \bar{C}_D) = f\left(\frac{z}{c},\ R_N,\ M,\ \frac{\delta_1}{S},\ H_1,\ \frac{\xi_1\delta_1}{U_1},\ \frac{l}{c},\ \frac{S}{c},\ C_L\right) \qquad (55)$$

Here the lift coefficient (or other blade-loading parameter) represents the most important effect of flow geometry. We would expect such a local loss coefficient to be almost independent of l/c unless δ/l is large, when the secondary flow on one end wall influences the flow near the other end wall. The averaged loss coefficient may be obtained as an area average $\zeta_{av}=(c/l)\int_0^l\bar{\zeta}d(z/c)$ (or as a mass-averaged loss as Dunham 1970 prefers). We would expect ζ_{av} to vary inversely as (l/c) for large aspect ratio.

Let us consider the effect of the independent variables on ζ_{av}.

(1) Hubert (1963) concluded from his experiments in a cascade, with R_N varying from 10^5 to 10^6, that the effect of Reynolds number on secondary flow and losses is negligibly small.

(2) Compressibility effects were studied experimentally by Hebbel (1965) and Namba & Asanuma (1967). Hebbel's experiments were carried out in a turbine cascade, whereas Namba's data were obtained from a transonic compressor cascade with artificially generated Mach-number profiles at inlet. The major conclusions of these two investigators are (i) that secondary-flow losses hardly change at moderate Mach numbers [it appears that if Mach-number effects are allowed for in calculating C_L, Mach-number dependency can be removed from the functional relationship (equation [55]) at moderate Mach numbers], and (ii) that the strength and structure of the shock waves show fundamental changes from the tests with uniform inlet flow, beyond the critical Mach number.

(3) Perhaps the most important flow parameters are some measure of the inlet velocity profile (H_1 or du_1/dz), the shear-layer or boundary-layer thickness δ_1/S, and a measure of the skewing of the boundary layer such as $\xi_1\delta_1/U_1$. Wolf (1959) carried out experiments in a turbine cascade with different inlet boundary-layer thicknesses (with no skew) but approximately the same shape factor H_1. The authors have noticed (by integrating Wolf's local loss coefficients) that there is a critical δ_1/S beyond which the losses decrease. This is similar to the trend predicted by Hawthorne (1955), who found that the *kinetic energy* in secondary flow is strongly dependent on δ_1/S, with all other parameters held constant. However, viscous effects, especially the losses due to wall stall and direct frictional losses in the end-wall region, should be more dependent on δ_1/c, which from equation (42) appears to control the Bernoulli-surface rotation ($\pi/2-\beta\simeq\theta_c c/2\delta$, for a linear profile). Hence the choice of δ_1/S or δ_1/c as a controlling parameter is still open to debate, although some investigators, notably Dunham (1970), have used the parameter δ_1/c instead of δ_1/S. Very little information is available on the effect of

H_1 on secondary losses or of the entry skew $\xi_1\delta_1/U_1$ although the latter must have a dominant effect.

(4) There is still some controversy regarding the effect of aspect ratio (l/c). Basing his opinion on much experimental data, Dunham (1970) concludes that the integrated losses *are* inversely proportional to aspect ratio. This is to some extent confirmed by Hawthorne's (1955) expression for the kinetic energy in secondary flow,

$$e = \frac{\theta_c^2(S/c)^2 \cos^2 \alpha_2}{l/c} f\left(\frac{\delta}{S'}\right) \tag{56}$$

which is valid for a linear inlet velocity profile.

(5) It is now firmly established that the secondary-flow losses vary directly as C_L^2 (Dunham 1970).

Dunham has concluded from a survey of all known experiments that the loss can be tentatively represented by the correlation

$$\zeta_{av} = \frac{c}{2l} \frac{\cos \alpha_2}{\cos \beta_1} \frac{C_L^2}{(S/c)^2} \frac{\cos^2 \alpha_2}{\cos^3 \alpha_m}\left(0.0055 + 0.078\sqrt{\frac{\delta_1}{c}}\right) \tag{57}$$

where β_1 is the blade inlet angle and α_m is the cascade mean angle. As discussed earlier, the use of δ_1/c alone (instead of δ_1/S' in addition) is open to debate.

APPLICATION IN PREDICTING TURBOMACHINERY PERFORMANCE

How is the designer of an axial flow compressor or turbine to use the mass of analytical and experimental work that has been done on secondary flow? He can use the various correlations of secondary loss discussed by Dunham (1970) together with empirical corrections for angle changes (such as the "work done" correction factor of Howell 1942), as most designers do at present. However, these empirical rules take but little account of the major developments in the understanding of secondary flow.

Achievements

Before suggesting alternative approaches to turbomachinery design using secondary-flow theory it is perhaps as well to summarize what has been achieved, and where secondary-flow analysis is still in an unsatisfactory state.

(1) The growth of secondary vorticity in channels can be calculated accurately for weak shear flows and in regions where viscous effects are small.

(2) Induced secondary velocities may be calculated reasonably well, in both plane and twisted blade rows, if no separation (usually in the corner between annulus wall and suction surface) takes place.

(3) The interaction of the secondary flow within the channel with the vortex sheets trailing from the channel walls (or blades) is well understood. These vortex

sheets contain both shed vorticity (arising from change in circulation along the blades) and trailing filament vorticity (arising from stretching of the entering vortex filaments).

(4) The growth of the secondary vorticity in rotating channels can be assessed. It is the absolute vorticity resolved in the direction of the relative velocity that is required for the calculation of the secondary velocities.

(5) In reading the literature, one should realize that secondary vorticity is defined differently by two of the most important contributors to this field. While the Hawthorne and A. G. Smith equations (5) and (16) give the *total* streamwise vorticity within a channel, the L. H. Smith equations (30) and (32) give the "excess" secondary vorticity—the vorticity over and above the primary vorticity that is calculated on the basis of closely spaced blading.

(6) As an alternative to the small-shear, large-disturbance analysis, the lifting-surface theory due to Honda and Namba can be employed to predict the variation in lift coefficient and the induced drag due to secondary flow. It should be noted here that the analysis is limited to small turning.

Limitations

(1) For secondary flow associated with wall boundary layers, some form of boundary-layer calculation method must be used to give the normal vorticity (η_2) in order that the secondary vorticity may be calculated from the Hawthorne–A. G. Smith equations; this calculation is also essential even for the Honda–Namba equations used in airfoil theory.

(2) Presence of a corner stall effectively prevents the calculation of secondary velocities.

(3) Secondary vorticity arising from one row has a major effect on the secondary velocities in the next row. Errors in predicting the secondary flow become cumulative as a result, in a multistage machine. However, it is this very effect that may reduce the skewing of wall boundary layers in multistage machines and the extent of the corner stall between suction surface and annulus wall.

The Present Position

With these achievements and limitations in mind, it should be possible for a designer to make a good estimate of the boundary-layer growth in the first two or three rows of a multistage machine and to estimate the growth of secondary vorticity and the corresponding secondary vorticities and changes in outlet flow angles (absolute for stationary rows, relative for rotating rows). The Hawthorne–A. G. Smith equations, or the L. H. Smith equations, can be used for calculating the secondary vorticity once the normal vorticity growth has been estimated from a boundary-layer analysis; the secondary velocities are best calculated from the Hawthorne–Novak approach for twisted blades; and care must be taken to allow for the transfer of streamwise vorticity at exit from one row to entry at the next.

Beyond the first two or three rows we cannot expect detailed secondary-flow analysis to be of much use, as the errors are cumulative, and we must expect empirical secondary-loss correlations to serve, if rather unsatisfactorily, for many years.

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Literature Cited

Balje, O. E. 1968. *Trans. ASME A* 90:309–60

Dean, R. C. 1954. ScD Thesis. MIT

Dixon, S. L. 1972. *Fluid mechanics and design of turbomachines*, ed. B. Lakshminarayana, W. R. Britsch, W. Gearhart. NASA SP304. In press

Dring, R. P. 1971. *ASME paper 71-GT-6*

Dunham, J. 1970. *J. Mech. Eng. Sci.* 12:49

Ehrich, F. F. 1955. *J. Aeronaut. Sci.* 22:51

Ehrich, F. F., Detra, R. W. 1954. *J. Aeronaut. Sci.* 21:136

Gregory-Smith, D. G. 1970. *Trans. ASME A* 92:369

Hawthorne, W. R. 1951. *Proc. Roy. Soc. London Ser. A* 206:374

Hawthorne, W. R. 1954. *J. Aeronaut. Sci.* 21:588

Hawthorne, W. R. 1955. *British ARC Rep.* 17, 519

Hawthorne, W. R. 1961. *Proc. Seminar Aerosp. Sci. Bangalore (India), 305*

Hawthorne, W. R. 1965. *Research Frontiers in Fluid Dynamics*, ed R. J. Seeger, G. Temple, Chap 1, 1–20. New York: Interscience

Hawthorne, W. R. 1966. *Gas Turbine Lab. Rep. No. 88*, MIT

Hawthorne, W. R. 1967. *Fluid Mechanics of Internal Flow*, ed. G. Sovran, 263. Amsterdam: Elsevier

Hawthorne, W. R., Armstrong, W. D. 1955. *Quart. J. Mech. Appl. Math.* 8:280

Hawthorne, W. R., Armstrong, W. D. 1956. *Aeronaut. Quart.* 7:247

Hawthorne, W. R., Novak, R. A. 1969. *Ann. Rev. Fluid Mech.* 1:341

Hebbel, H. H. 1965. *Z. Flugwiss* 13:324

Honda, M. 1961. *Proc. Roy. Soc. London Ser. A* 265:46

Horlock, J. H. 1963. *Trans. ASME D* 85:55

Horlock, J. H. 1966. *Axial Flow Turbines*, 102. London: Butterworth

Horlock, J. H. 1971. *Cambridge Univ. Rep. CUED/A-Turbo/TR 28*

Horlock. J. H. et al. 1966. *Trans. ASME D* 88:637

Horlock, J. H., Hoadley, D. 1970. *British ARC* CP 1196

Horlock, J. H., Lakshminarayana, B. 1973. To be published

Howell, A. R. 1942. *British ARC R & M 2095*

Hubert, G. 1963. *Verein Deutsches Ingenieurs Forsch.* 496

Johnston, J. P. 1960. *Trans. ASME D* 82:233

Joubert, P. et al. 1967. *Fluid Mechanics of Internal Flow*, ed. G. Sovran. Amsterdam: Elsevier

von Kármán, Th., Tsien, H. S. 1945. *Quart. Appl. Math* 3:1

Lakshminarayana, B. 1970. *Trans. ASME D* 92:467

Lakshminarayana, B., Horlock, J. H. 1963. *Int. J. Mech. Sci.* 5:287

Lakshminarayana, B., Horlock, J. H. 1967a. *Trans. ASME D* 89:191

Lakshminarayana, B., Horlock, J. H. 1967b. *British ARC R & M 3483*

Lighthill, M. J. 1956. *J. Fluid Mech.* 1:31

Loos, H. G. 1953. *Guggenheim Propulsion Lab. CIT Rep. 3*, California Inst. Technol.

Loos, H. G. 1956. *J. Aeronaut. Sci.* 23:76

Louis, J. F. 1956. *Proc. 9th Int. Congr. Appl. Mech.* 3:306

Marris, A. W. 1963. *Trans. ASME E* 30:525

Marris, A. W. 1966. *Trans. ASME D* 88:537

Martin, M. E. 1959. *British ARC CP 425*

Mellor, G. L., Wood, G. M. 1970. *ASME Paper 70-GT-80*

Nally, M. C., Hawthorne, W. R. 1969. *Nat. Eng. Lab (Glasgow) Rep. 432*

Namba, M. 1969. *J. Fluid Mech.* 36:735

Namba, M., Asanuma, T. 1967. *Bull. JSME* 10:920

Smith, A. G. 1957. *Aeronaut. Quart.* 8:369

Smith, L. H. 1955. *Trans. ASME* 77:1065

Squire, H. B., Winter, K. G. 1951. *J. Aeronaut. Sci.* 18:271

Wolf, H. 1959. *Wiss. Z. Tech. Hochsch. Dresden* 8:763