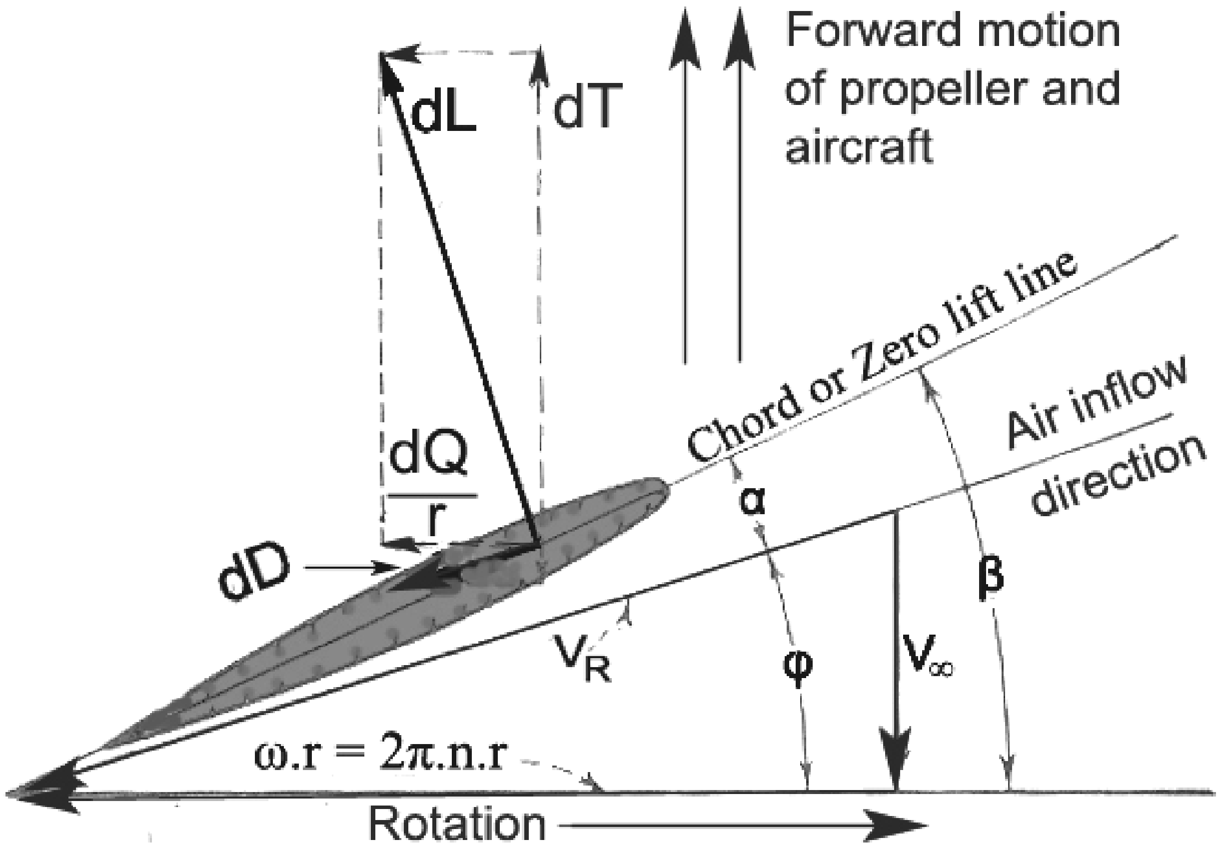
AE 658

Propeller Theory and Design

Combined Momentum and Blade Element Theory



Forward motion of propeller and aircraft
dL
dT
dQ
r
dD
Chord or Zero lift line
Air inflow direction
α
β
φ
V_R
V_∞
ω.r = 2π.n.r
Rotation

Extensions of Blade Element and Momentum Theories

Extension of BET

The drag of a propeller core may be written as:

$$D_1 = ½\, \rho V_1^2\, A_1 C_{D1}$$

Where A_1 is the frontal area of the propeller core.& C_{D1} is the drag of the core., $C_{D1} \approx 1$. This drag is often small compared to the thrust produced and hence is neglected. In case of small propellers or micro-propellers this term may be significant.

Thus the thrust coefficient becomes :

$$C_T = ½\, \dot{m}.\int_{r_1/d}^{1/2} [J^2 + (2.\pi.r/d)^2]\,(C_L \cos\phi - C_D \sin\phi).c/d.\, d(r/d) - ½\, J^2.A_1/d^2$$

where, C_{D1} = 1 assumed.

It is seen that C_T depends largely on C_L --- as C_D is small and sin α is also small

Propeller torque, Q = $\dot{m}.\int$ r. *d*F$_t$, and

power, P = ω.Q = 2.π.n. $\int$ r. *d*F$_t$

Propeller Power,

$$P = 2.\pi.n.\tfrac{1}{2}\,\rho.n^2d^2m.\int_0^{d/2}[J^2 + (2.\pi.r/d)^2](C_L \sin\phi + C_D \cos\phi).r.c.dr$$

In the core region between r = 0 to r = r$_1$ may be neglected for all practical purposes. Hence,

$$C_P = .\int_{r_1/d}^{1/2}[J^2 + (2.\pi.r/d)^2](C_L \sin\phi + C_D \cos\phi).r/d.c/d.\,d(r/d)$$

Thus C$_P$ is a function of J , advance ratio for a given propeller shape i.e. C$_L$ and C$_D$

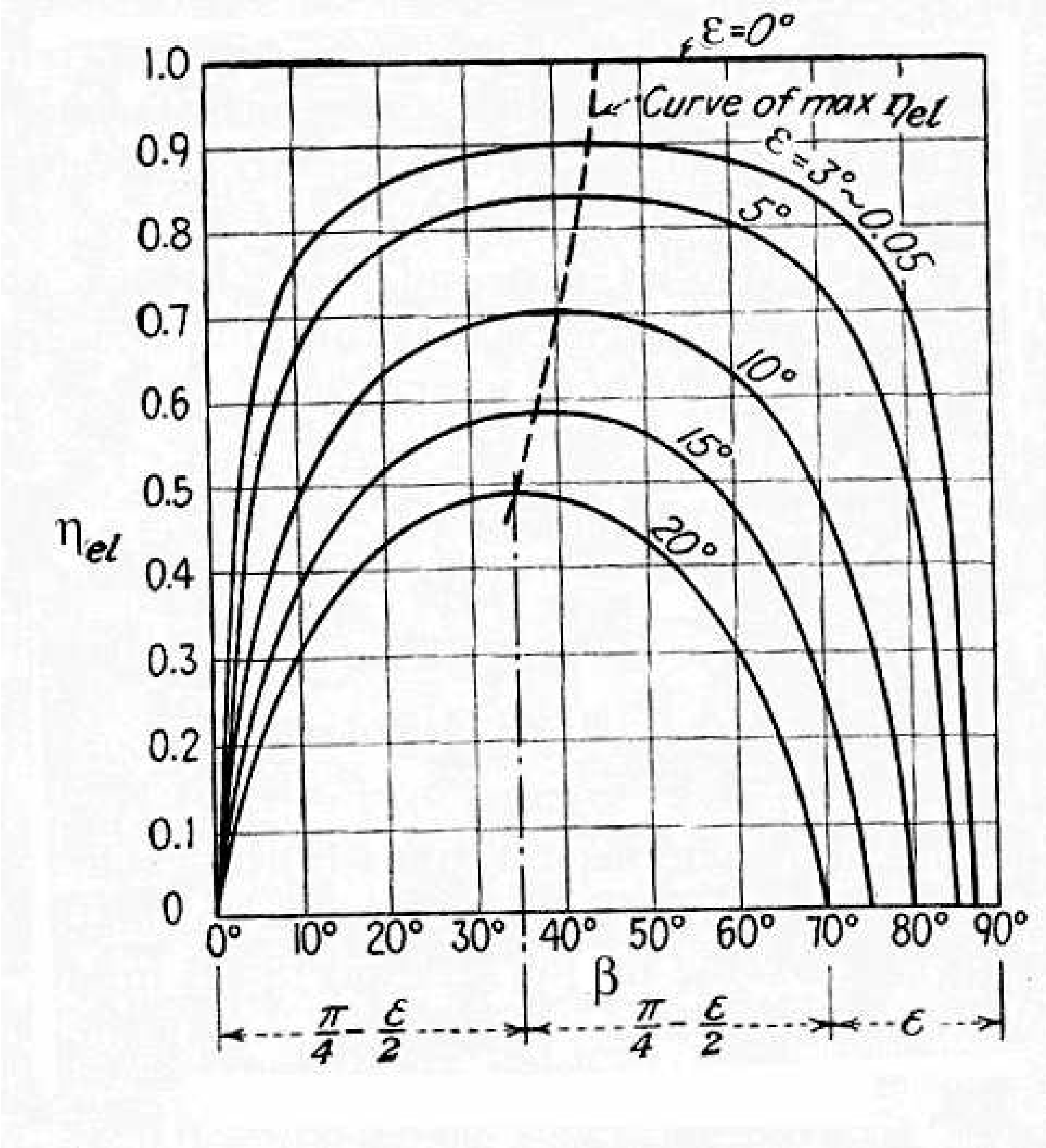
The above equations for thrust and power coefficients may be replaced by replacing the factor

$$[J^2 + (2.\pi.r/d)^2] = J^2 (1 + \cot^2 \phi) = J^2 / \sin^2\phi$$

$$C_T = J^2/2d^2 . \{ m \int_{r_1}^{d/2} [(C_L \cot \phi - C_D) /\sin \phi] c. \, dr - A_1 \}$$

$$C_P = \pi m . J^2/d^2 \int_{r_1}^{d/2} [(C_L + C_D. \cot \phi) \sin \theta]. c. r. \, dr$$

Now, the gliding angle ε is defined as $\tan \varepsilon = C_D / C_L$


ε=0°
Curve of max η_{el}
ε=3°~0.05
5°
10°
15°
20°
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
η_{el}
0° 10° 20° 30° 40° 50° 60° 70° 80° 90°
β
$\frac{\pi}{4}-\frac{\varepsilon}{2}$
$\frac{\pi}{4}-\frac{\varepsilon}{2}$
ε

The efficiency of the blade element may also be written down as

$$\eta_{el} = \tan \alpha / \tan (\alpha + \varepsilon)$$

The maximum efficiency may be given as

$$(\eta_{el})_{max} = \frac{\tan\left(\frac{\pi}{4} - \frac{\epsilon}{2}\right)}{\tan\left(\frac{\pi}{4} + \frac{\epsilon}{2}\right)} = \frac{\left(1 - \tan\frac{\epsilon}{2}\right)^2}{\left(1 + \tan\frac{\epsilon}{2}\right)^2} \sim \frac{1 - \epsilon}{1 + \epsilon} \sim 1 - 2\epsilon$$

The last two expressions are valid for small ε only

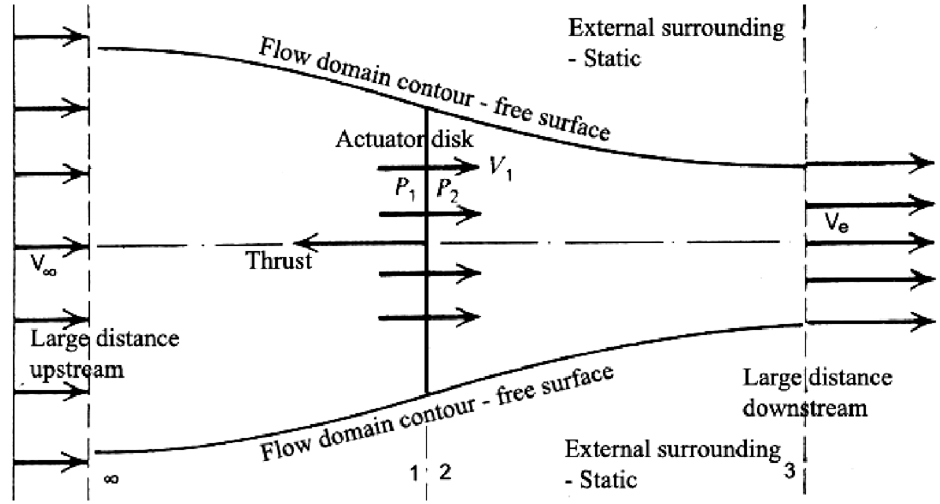
The efficiency plot shown above shows for $\varepsilon = 0$ we get, $\eta_{el} = 1$ independent of β. It can be proved that, as visible from the plot that the curves are symmetrical with respect to the line, $\beta = \pi/4 - \varepsilon/2$.

For $\beta = 0$ and $\beta = \pi/2 - \varepsilon$ we have $\eta_{el} = 0$

$$\eta_{prop} = \left[\int \eta_{el} \cdot dP \right] / \int dP$$

The propeller efficiency would assume a value of 1 for $\varepsilon = 0$ which means drag coeff. is 0. The maximum elemental efficiency that the blade can assume is $(1 - 2.\varepsilon_{min})$ where εmin denotes the minimum ε occurring on the blade. Since this maximum elemental efficiency can occur only at one blade efficiency the overall propeller efficiency will always be less than the maximum elemental efficiency as long as all *d*P are positive . For any $\varepsilon < \varepsilon_{max}$ the η_{el} lies above the efficiency curve of ε_{max} .

Extension of Momentum Theory



If one applies change of velocity component in all three directions i.e. changes are designated v_1 , w_1 , u_1 - as the three components after the actuator disc. The energy equation can now be derived as upstream of the actuator disc :

$$\rho/2.\ v^2 + p_\alpha = \rho/2.[\ (V+v)^2 + w^2\ ]+ p_1$$ **------------ upstream equation**

$$\rho/2.\ [(V + v)^2 + w^2 + u^2\ ] + p_2 = \rho/2.\ [\ (V_1^2 + v_1^2) + w_1^2 + u_1^2\ ] + p_e$$

------- Downstream

Where one may write $\mathbf{p_2 = p_1 + p'}$ **over the disk cross section**

The velocity components v, w, u are small and their second order terms and the differences of the second order terms are negligible compared to those of the main velocity components.

Now, if energy equation is applied between the stations α-α to 1-1 upstream of the actuator disc, we get :

$$\frac{p_1 - p_0}{\gamma} + \frac{(V^2 + v_1) + w_1^2 + u_1^2}{2g} - \frac{V_1^2}{2g} = \frac{p_1 - p_0}{\gamma} + \frac{v_1(2V + v_1)^2 + w_1^2 + v_1^2}{2g}$$

In the above equation w_1^2 is of the fourth order and $u_1^2 \approx u^2$ and their difference is of the third order. The basic equations of the Momentum theory established earlier are not sufficient or intended to determine the unknown functions v, v_1, u at the radius r. To determine these unknowns it would be necessary to set up the differential equations governing the motion of the fluid and to solve them on the basis of a given propeller shape.

The simplest set of plausible assumptions is the following : (1) The axial velocity components v and v_1 are assumed to be constant over the entire cross sections. (2) The tangential velocity components u of the immediate downstream of the actuator disk are assumed to be proportional to the distance r from the propeller axis $u = r.\omega'$. The assumptions leads to v, v_1 and u/r by certain average values. The first assumption leads to $v = \frac{1}{2}v_1$ by virtue of earlier theory. Note that v_1 and ω' are the axial and tangential values imparted by the actuator disc on the fluid.

Accordingly we can write :

$$\int u^2\, dS = 2\pi \int_0^{d/2} u^2 r\, dr = 2\pi\omega'^2 \int_0^{d/2} r^3\, dr = 2\pi\omega'^2 \frac{1}{4}\left(\frac{d}{2}\right)^4 = \frac{\omega'^2 d^2}{8}$$

$$\int ru\, dS = 2\pi \int_0^{d/2} r^2 u\, dr = 2\pi\omega' \int_0^{d/2} r^3\, dr = \frac{\omega' d^2}{8} S$$

Thus, thrust as per momentum theory may be written as :

$$T = \rho S\left(V + \frac{v_1}{2}\right)v_1$$

And, power put in by the actuator disk is :

$$P = \rho S\left(V + \frac{v_1}{2}\right)\omega\omega'\frac{d^2}{8}$$

$$P = \rho S\left(V + \frac{v_1}{2}\right)\left[\left(V + \frac{v_1}{2}\right)v_1 + \omega'^2\frac{d^2}{16}\right]$$

Or, $$P = T\left(V + \frac{v_1}{2}\right) + P\frac{\omega'}{2\omega}$$

Thus the (induced) efficiency of a propeller as ADT is :

$$\eta_i = \frac{TV}{P} = \frac{1 - \omega'/2\omega}{1 + v_1/2V}$$

Or,

$$\eta_i = \frac{TV}{P} = \frac{8Vv_1}{\omega\omega' d^2} = \frac{4Jv_1}{\pi\omega' d} = \frac{2}{\pi^2} J^2 \frac{v_1/V}{\omega'/\omega}$$

A set of new Thrust Loading and Power Loading are defined as :

$$\tau = \frac{2T}{\rho V^2 S}, \qquad \sigma = \frac{2P}{\rho V^3 S}$$

The parameters C_T and C_P are connected with τ and σ by

$$C_T = \frac{T}{\rho n^2 d^4} = \frac{TJ^2}{\rho V^2 d^2} = \frac{\pi}{8} J^2 \tau$$

$$C_P = \frac{P}{\rho n^3 d^5} = \frac{PJ^3}{\rho V^3 d^2} = \frac{\pi}{8} J^3 \sigma$$

Note that the ratio τ/σ equals the induced efficiency

$$\eta_i = \frac{VT}{P} = \frac{JC_T}{C_P} = \frac{\tau}{\sigma}$$

Introducing the dimensionless parameters τ and σ, Eqs. can be written as

$$\tau = 2\left(1 + \frac{v_1}{2V}\right)\frac{v_1}{V}$$

$$\sigma = 2\left(1 + \frac{v_1}{2V}\right)\frac{\pi^2}{2J^2}\frac{\omega'}{\omega}$$

$$\sigma = 2\left(1 + \frac{v_1}{2V}\right)\left[\left(1 + \frac{v_1}{2V}\right)\frac{v_1}{V} + \frac{\pi^2}{4J^2}\frac{\omega'^2}{\omega^2}\right] = \tau\left(1 + \frac{v_1}{2V}\right) + \sigma\frac{\omega'}{2\omega}$$

Exit induced velocity *v* may be found from :

$$\left(\frac{v_1}{V}\right)^2 + 2\frac{v_1}{V} = \tau \qquad \text{or} \qquad \frac{v_1}{V} = \sqrt{\tau + 1} - 1$$

The induced whirl speed ω' may be found from :

$$\frac{\omega'}{\omega} = \frac{J^2}{\pi^2} \frac{\sigma}{1 + v_1/2V} = \frac{2J^2\sigma}{\pi^2(\sqrt{\tau + 1} + 1)} = \frac{2J^2}{\pi^2} \frac{\sigma}{\tau} (\sqrt{\tau + 1} - 1)$$

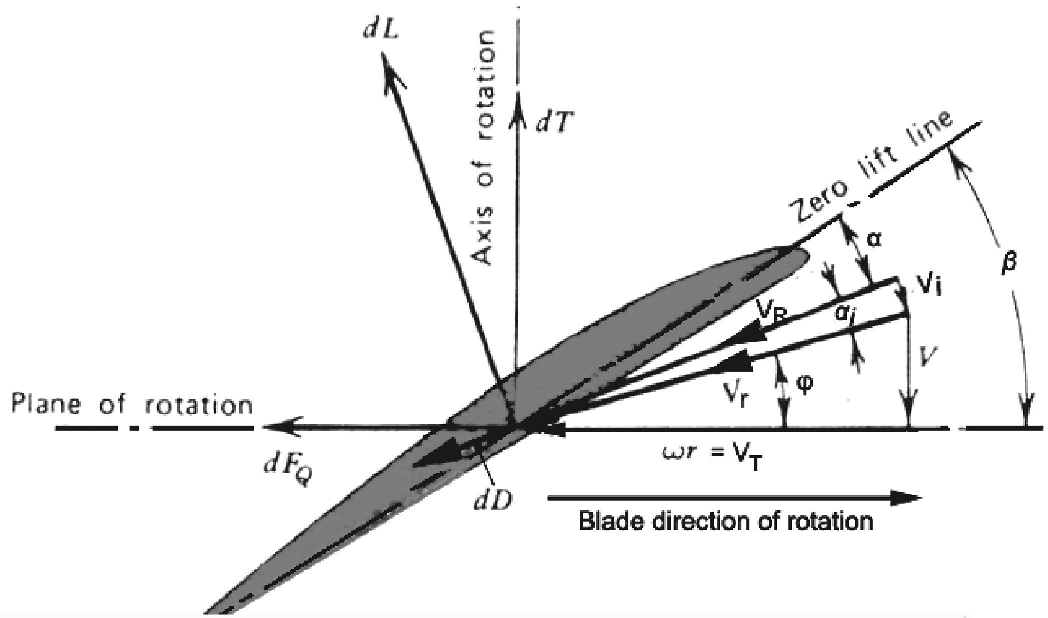
A unique relationship between the thrust, and torque loading and efficiency can now be written down as :

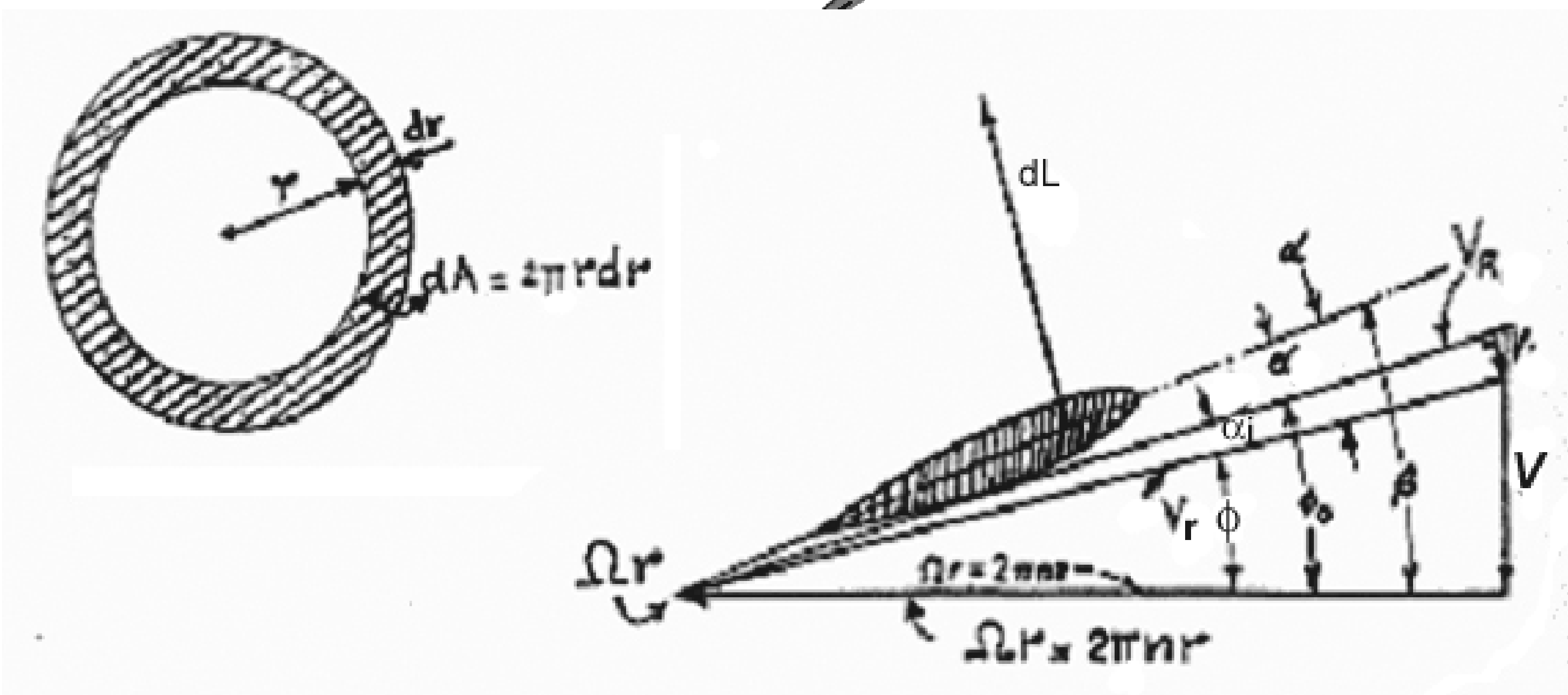
$$\frac{\sigma}{\tau} = \frac{1}{\eta_i} = \tfrac{1}{2}(\sqrt{\tau + 1} + 1) + \frac{J^2}{\pi^2} \frac{1}{\eta_i^2} (\sqrt{\tau + 1} - 1)$$

As a first approximation the relation between τ and η may be written as :

$$\eta_i = \frac{2}{\sqrt{\tau + 1} + 1}$$

Combined Blade Element and Momentum Theory





In this theory, the induced flow due to lift production on a blade is actually computed. For this purpose, it is noted that in a momentum theory, thrust can be expressed in terms of induced velocity. If the number of blades is B, the total elemental thrust from simple blade element theory (omitting term) is:

$$B.dT = B.c.L.\cos\phi_0 = B.C_l.\frac{1}{2}\rho.V_R{}^2 c.dr.\cos\phi_0$$

where ϕ_0 is defined in fig.. Note that the induced velocity in the thrust direction is $v_i.\cos\phi_0$

Therefore, by momentum theory, the total elemental thrust is also given by

$$B.dT = \rho.(2.\pi.r.dr).(V + v_i\cos\phi_0).(2v_i\cos\phi_0)$$

Equating the two equations we find that:

$$v_i = \frac{B.C_l.c.V_R^2}{8.\pi.r\left(V_r + v_i \cos\phi_0\right)}$$

To simplify it is assumed that the angle α_ι is very small.

Hence:

$$\sin\phi_0 = \sin(\phi + \alpha_i) \approx \sin\phi + \alpha_i.\cos\phi$$

$$= \frac{V_r + v_i \cos\phi_0}{V_R}$$

Using $\tan\alpha_i = v_i / V_R \approx \alpha_i$ and $C_l = a_0(\beta - \phi - \alpha_i)$

We get, $$\frac{V_r}{V_R} = \alpha_i \cong \frac{B.c}{8\pi r}.\frac{a_0(\beta - \alpha_\iota - \alpha_o)}{\sin\alpha_i + \alpha_{i.}\cos\alpha_i}$$

Define the solidity ratio, as the ratio of total bladed area to the disk (swept) area:

$$\sigma = \frac{B.c.R}{\pi R^2} = \frac{B.c}{\pi R} \qquad \text{and} \qquad x = \frac{r}{R}$$

A quadratic equation is arrived at for α_i

$$\alpha_i^2 \cos\phi + (\sin\alpha_i + \frac{a_0.\sigma}{8x}).\alpha_i - \frac{a_0\sigma}{8x}(\beta - \alpha_i) = 0$$

The solution of which is:

$$\alpha_i = \frac{1}{2\cos\theta}\left\{-(\sin\alpha_i + \frac{a_0\sigma}{8x}) + \sqrt{(\sin\alpha_i + \frac{a_0\sigma}{8x})^2 + 4\cos\alpha_i \frac{a_0\sigma}{8x}(\beta - \alpha_i)}\right\}$$

If the following approximations are used :

$$\cos\alpha_i \approx 1.0 \qquad \sin\alpha_i = V_r / V_t.x \qquad V_r \approx V_t.x$$

where V_t is the tangential velocity at the tip and x = r/R

the induced velocity $v_i = \alpha_i.V_t.x$ can be found from the eqn.s :

$$v_i = V_t\left\{-\left(\frac{V}{2V_t}+\frac{a_0\sigma}{16}\right)+\sqrt{\left(\frac{V}{2V_t}+\frac{a_0\sigma}{16}\right)^2+\frac{a_0\sigma}{8.x}(\beta-\alpha_i)}\right\}$$

It is used frequently to evaluate the induced velocity on helicopter rotors in vertical climbing flight.

Under low thrust conditions,

$$\alpha_i \cong \frac{\beta-\alpha_i}{1+\frac{8.x.\sin\alpha_i}{a_0\sigma}}$$

Simplifying for α_i

$$\boxed{\alpha_i = \frac{\beta-\phi}{1+\frac{8.x.\sin\phi}{\sigma.a}}}$$

The thrust and the torque gradients are finally given as : (<u>ref : Revised BET</u>)

$$\frac{dC_T}{dx} = 3.88x^2.\sigma.\psi_T$$

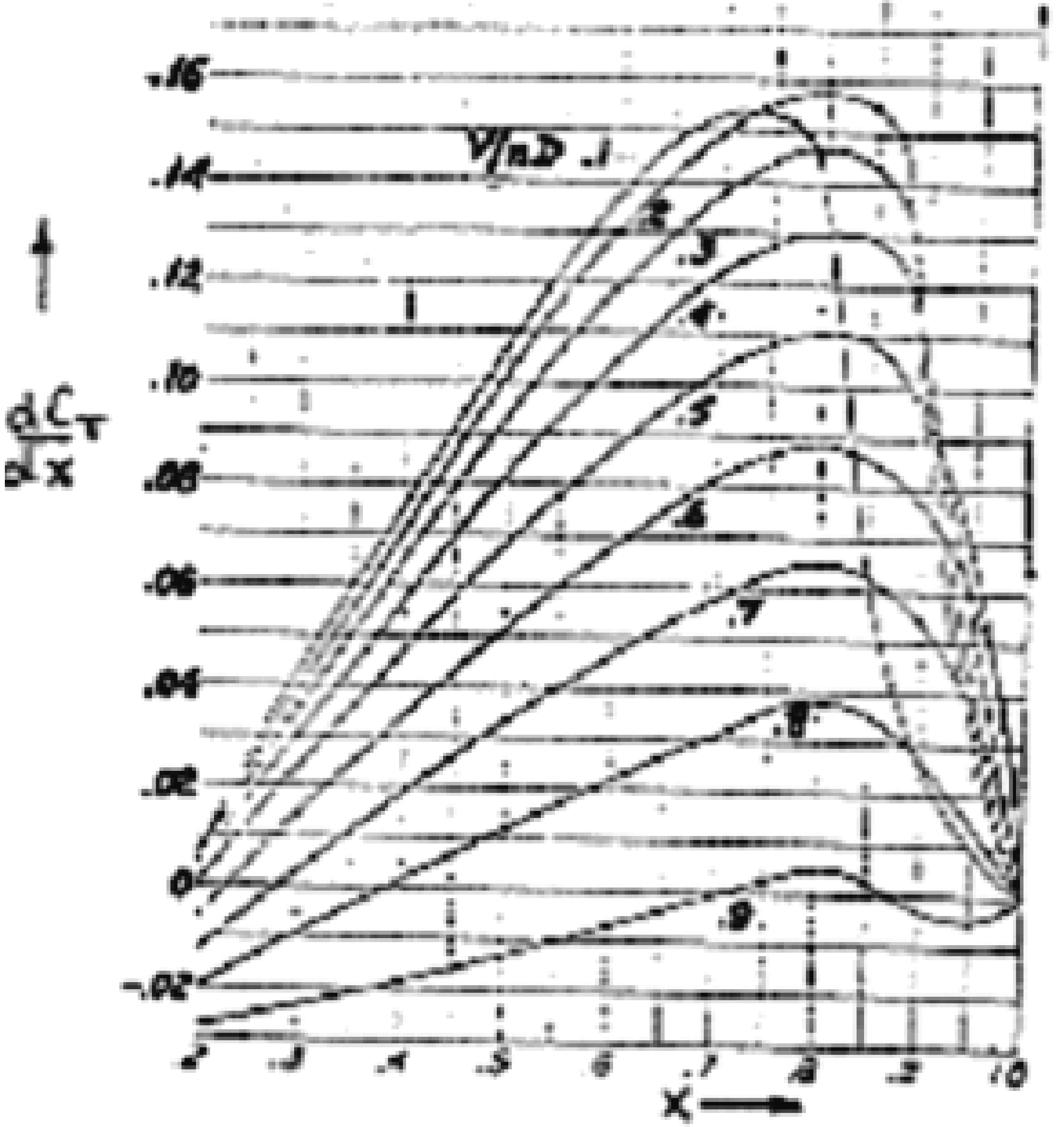
$$\frac{dC_Q}{dx} = 1.94x^3.\sigma.\psi_Q$$

Where,

$$\psi_T = \frac{\cos^2\alpha_i}{\cos^2\phi}(c_l\cos\phi_o - c_d\sin\phi_0)$$

$$\psi_Q = \frac{\cos^2\alpha_i}{\cos^2\phi}(c_l\cos\phi_o + c_d\sin\phi_0)$$

$$\boxed{\phi_0 = \phi + \alpha_i}$$



V/nD .1
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-.02
$\frac{dC_T}{dx}$
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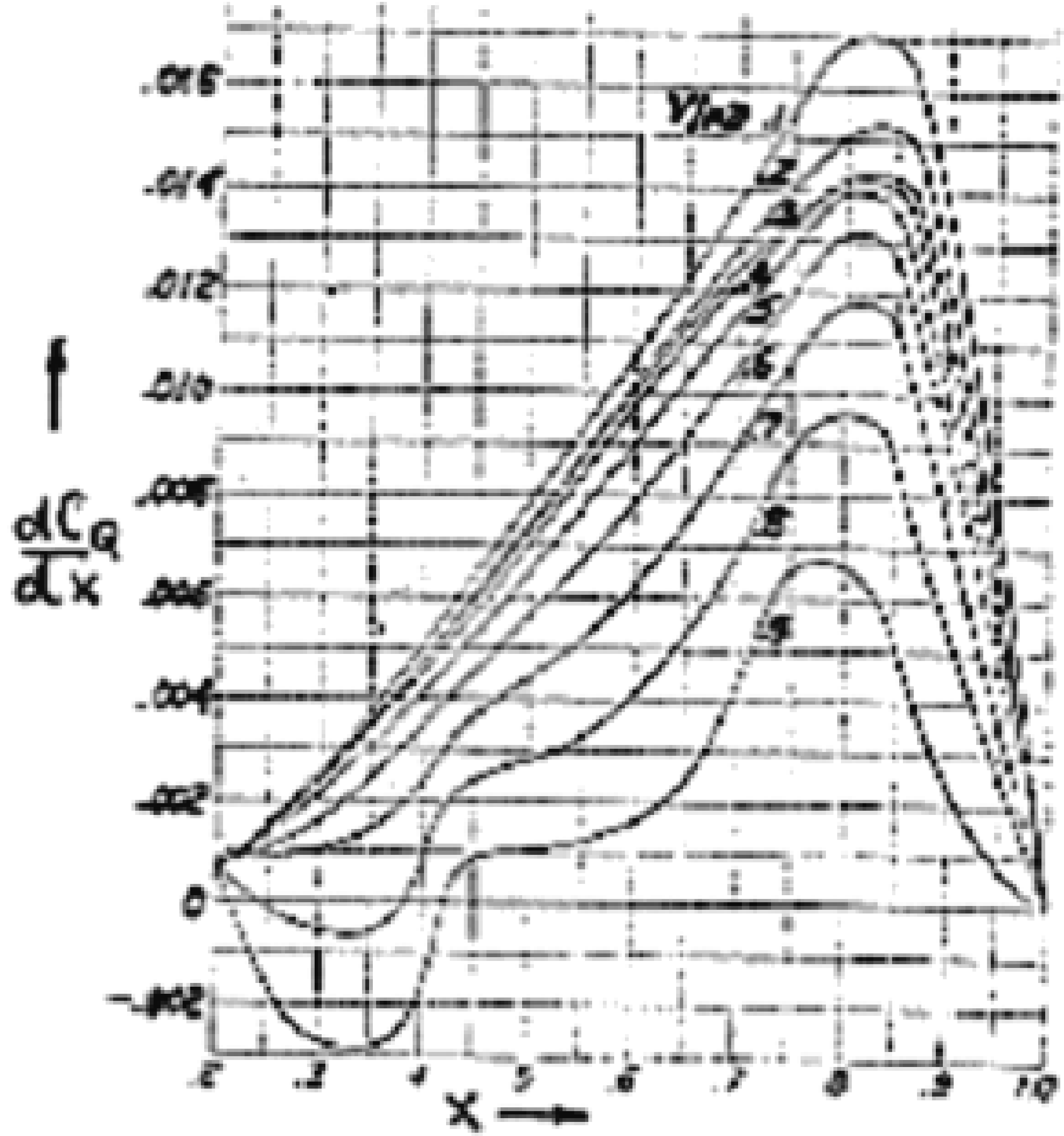
x


Y/b/2 .1
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0
-.002
$\frac{dC_Q}{dx}$
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.9
1.0
X

End of Propeller Chapter