ME 724: Turbulent Channel Flow

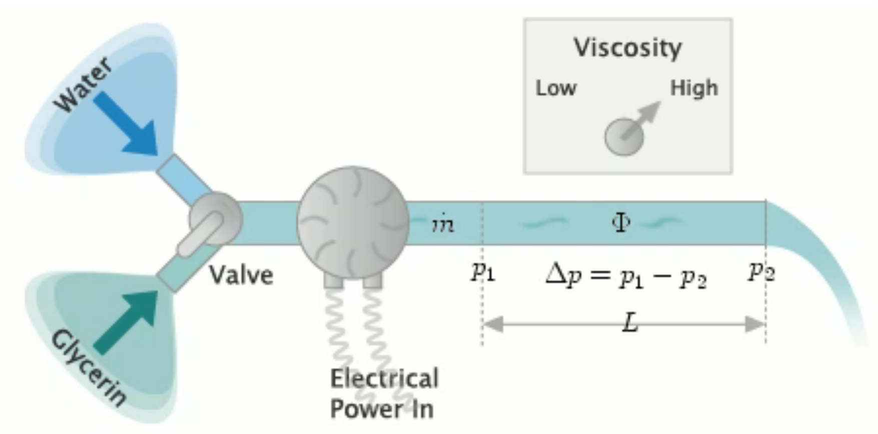
Instructor: Amitabh Bhattacharya

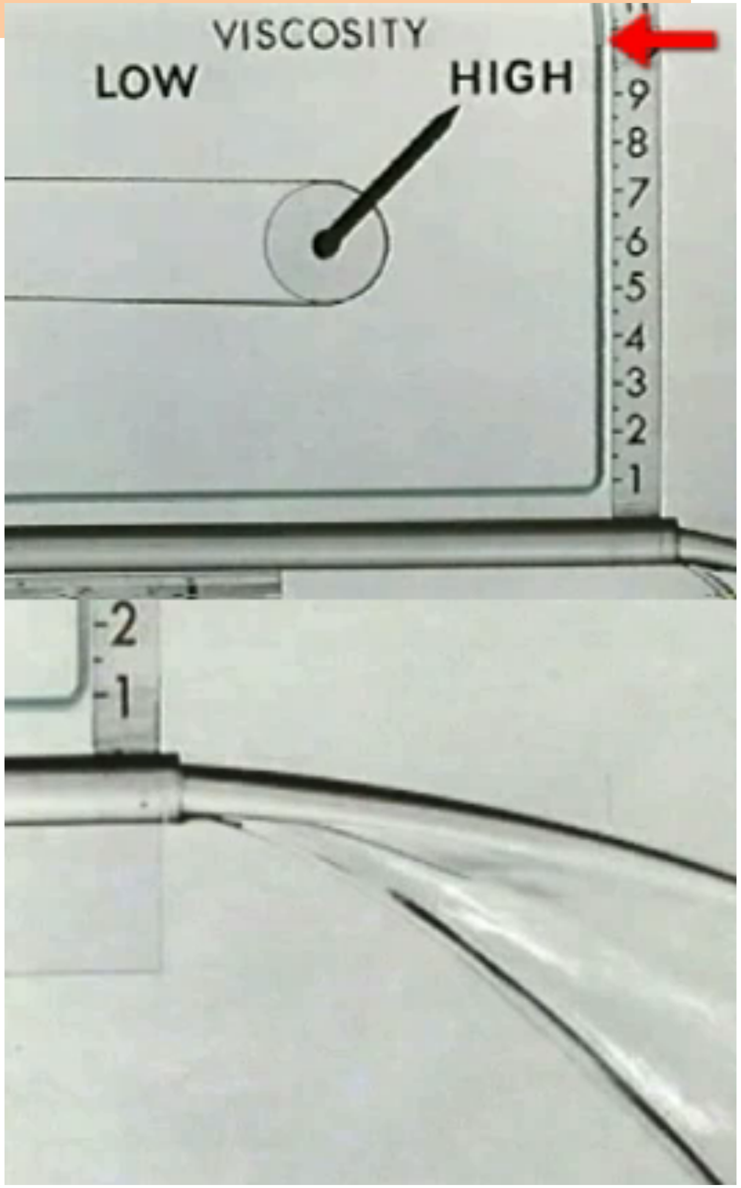
Solving for Channel Flow

- Exact mean equations in *y* direction
- Important scales
- Global momentum balance
- Understand significance of “log-region”
- Obtain log-law from matched asymptotics
- Apply mixing-length hypothesis, obtain log-law
- Get friction factor relationship

Channel/Pipe Flow

- Pressure gradient required is much higher for turbulent flow
  - For same flow rate





Moody Chart For Pipes

$$\text{Friction Factor} = f = \frac{\Delta p D}{L \frac{1}{2} \rho U_b^2}$$

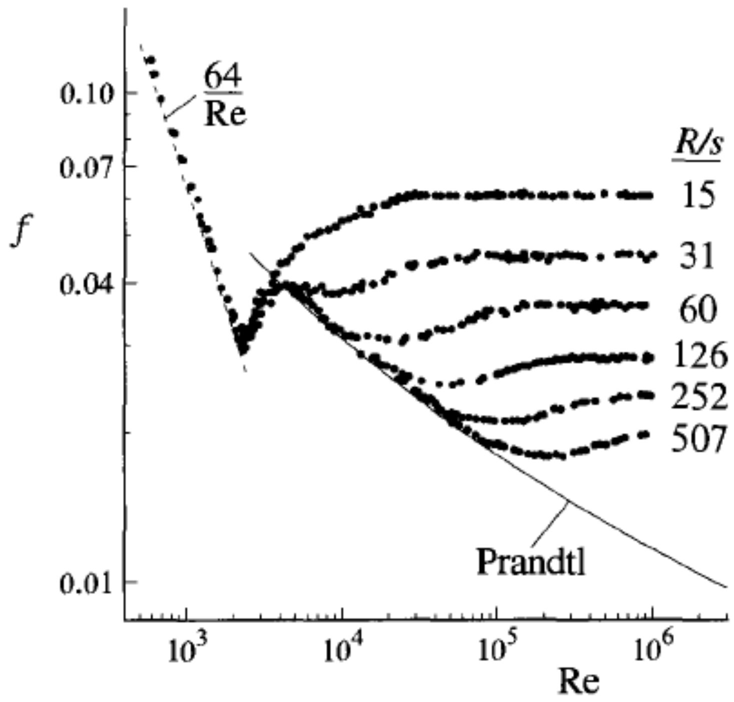
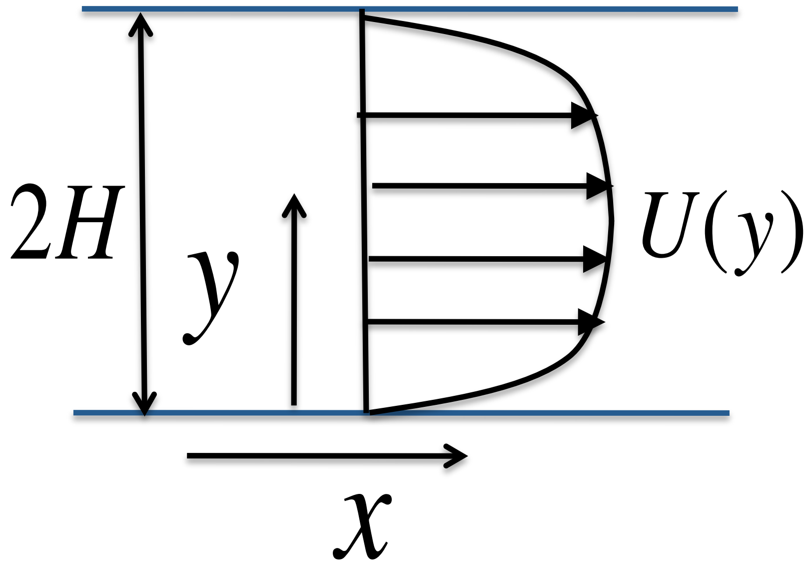


Fig. 7.23. The friction factor f against the Reynolds number for fully developed flow in pipes of various roughnesses. Dashed line, friction law for laminar flow; solid line, Prandtl friction law for turbulent flow in smooth pipes, Eq. (7.98); symbols, measurements of Nikuradse. (Adapted from Schlichting (1979) with permission of McGraw-Hill.)

Channel Flow

- **Aim:** Find pressure gradient required to pump turbulent flow at a given flow-rate



- Fully-developed flow

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial x} = \frac{\partial R_{ij}}{\partial x} = 0 \qquad \frac{\partial P}{\partial x} < 0 \ \ \text{(Drives the flow)}$$

- No-slip BC: $U(0) = U(2H) = V(0) = V(2H) = 0$

$$u'_i(0) = u'_i(2H) = 0 \Rightarrow R_{ij}(0) = R_{ij}(2H) = 0$$

Mean Equations in 2D

- Mean continuity eqn: $\cancel{\frac{\partial U}{\partial x}} + \frac{\partial V}{\partial y} = 0$

- Mean (steady) momentum equations

$$\cancel{U\frac{\partial U}{\partial x}} + V\frac{\partial U}{\partial y} = -\frac{\partial P}{\partial x} - \cancel{\frac{\partial R_{11}}{\partial x}} - \frac{\partial R_{12}}{\partial y} + \nu\left[\cancel{\frac{\partial^2 U}{\partial x^2}} + \frac{\partial^2 U}{\partial y^2}\right]$$

$$\cancel{U\frac{\partial V}{\partial x}} + V\frac{\partial V}{\partial y} = -\frac{\partial P}{\partial y} - \cancel{\frac{\partial R_{21}}{\partial x}} - \frac{\partial R_{22}}{\partial y} + \nu\left[\cancel{\frac{\partial^2 V}{\partial x^2}} + \frac{\partial^2 V}{\partial y^2}\right]$$

$$U = \overline{u}_1, V = \overline{u}_2,\ P = \overline{p},\quad R_{ij} = \overline{u'_i u'_j}$$

V Equation

- From continuity: $\partial V / \partial y = 0$
- No-slip :

$$V(0) = V(2H) = 0 \Rightarrow V = 0 \text{ for whole channel}$$

- Momentum eqn (y direction):

$$\frac{\partial P}{\partial y} = -\frac{\partial R_{22}}{\partial y} \quad \Rightarrow \quad \frac{\partial}{\partial y}\left[\frac{\partial P}{\partial x}\right] = 0 \quad \left(\because \frac{\partial R_{22}}{\partial x} = 0\right)$$

  – Mean pressure gradient is constant at all *x,y*

$$\frac{\partial P}{\partial x}(x, y) = \text{Constant}$$

U Equation

- U momentum eqn:

$$-\partial_x P = \frac{\mathrm{d}}{\mathrm{d}y}\left[R_{12} - \nu\frac{\mathrm{d}U}{\mathrm{d}y}\right]$$

- Global momentum balance

$$-\int_0^{2H} \partial_x P \,\mathrm{d}y = \int_0^{2H} \frac{\mathrm{d}}{\mathrm{d}y}\left[R_{12} - \nu\frac{\mathrm{d}U}{\mathrm{d}y}\right]\mathrm{d}y \quad \Rightarrow \quad -2H\partial_x P = \left[R_{12} - \nu\frac{\mathrm{d}U}{\mathrm{d}y}\right]\Bigg|_0^{2H}$$

$$-\partial_x P = \frac{\tau_{\mathrm{wall}}}{H}, \ \text{where,}\ \tau_{\mathrm{wall}} = \nu\left.\frac{\mathrm{d}U}{\mathrm{d}y}\right|_{y=0} = -\nu\left.\frac{\mathrm{d}U}{\mathrm{d}y}\right|_{y=2H}$$

- Mean pressure gradient and viscous wall stress balance each other

Scales in Channel Flow

- Momentum eqn simplifies to:

$$u_\tau^2 = H\frac{\mathrm{d}}{\mathrm{d}y}\left[R_{12} - \nu\frac{\mathrm{d}U}{\mathrm{d}y}\right] \Rightarrow R_{12} - \nu\frac{\mathrm{d}U}{\mathrm{d}y} = \frac{u_\tau^2 y}{H} - u_\tau^2$$

- Velocity scale (friction velocity): $u_\tau = \sqrt{\tau_\mathrm{w}}$
- 2 Length scales:

$$H\ (\text{"outer"})$$

or

$$l^+ = \nu / u_\tau\ (\text{"inner" or "+" units})$$

Reynolds vs Viscous Stress Variation with *y*

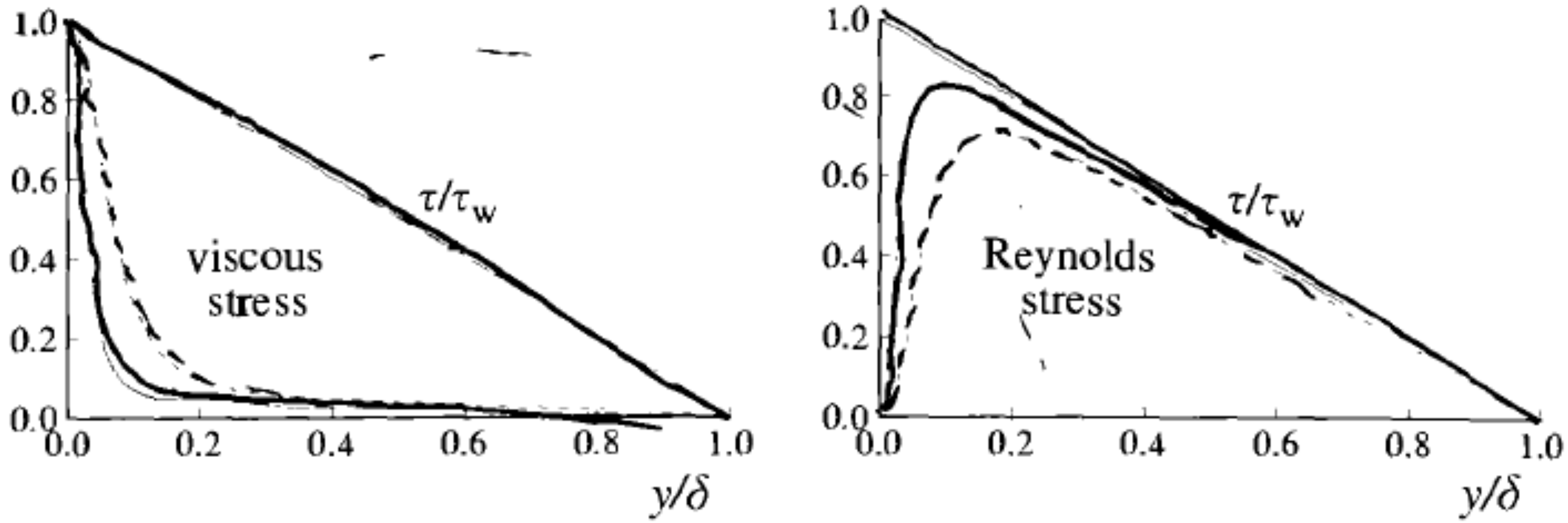


Fig. 7.3. Profiles of the viscous shear stress, and the Reynolds shear stress in turbulent channel flow: DNS data of Kim *et al.* (1987): dashed line, Re = 5,600; solid line, Re = 13,750.

$$R_{12} - \nu \frac{\mathrm{d}U}{\mathrm{d}y} = \tau_w \left[\frac{y}{H} - 1 \right]$$

Significance of Inner Length Scale

- Total stress (turbulent+viscous) at $\frac{y}{H} \to 0$:

$$R_{12} - \nu \frac{\mathrm{d}U}{\mathrm{d}y}$$

- From mixing length model:

$$R_{12} \sim y^2 \left(\frac{\partial U}{\partial y} \right)^2$$

- The stresses are comparable when:

$$y^2 \left(\frac{\partial U}{\partial y} \right)^2 \sim \nu \frac{\partial U}{\partial y} \sim \tau_w \Rightarrow \frac{y u_\tau}{\nu} = y^+ \sim 1$$

Reynolds vs Viscous Stress (Fractional Contributions)

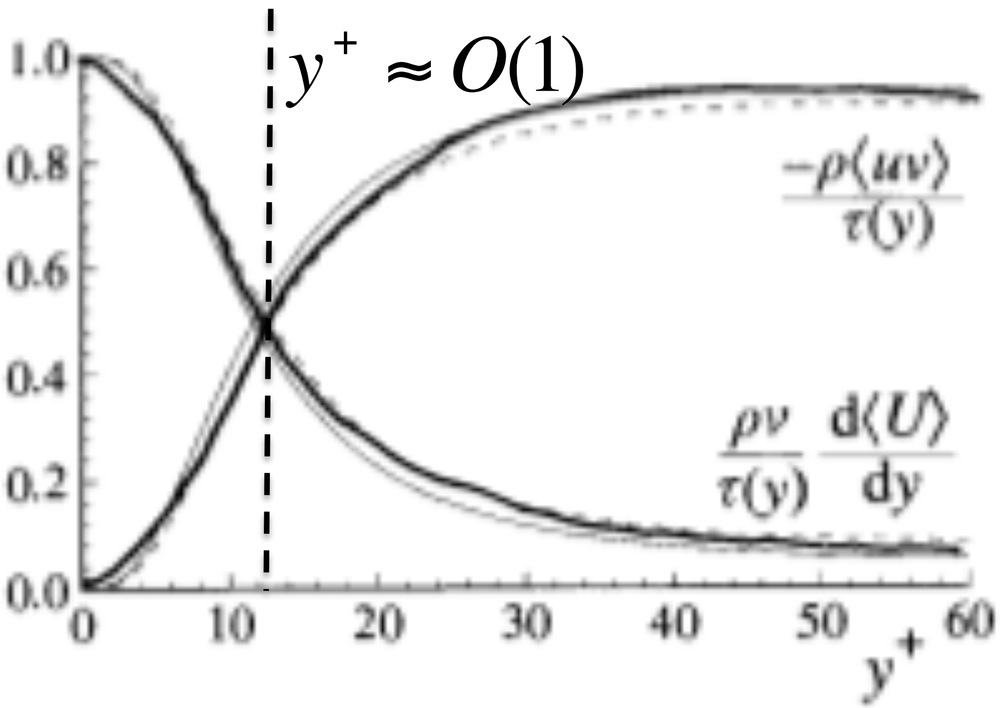


Fig. 7.4. Profiles of the fractional contributions of the viscous and Reynolds stresses to the total stress. DNS data of Kim *et al.* (1987): dashed lines, Re = 5,600; solid lines, Re = 13,750.

“Inner” and “Outer” Momentum Eqns

- 2 non-dimensional eqns possible:

$$R_{12}^{+} - \frac{\mathrm{d}U^{+}}{\mathrm{d}y^{+}} = \frac{y^{+}}{\mathrm{Re}_{\tau}} - 1 \qquad U^{+} = \frac{U}{u_{\tau}},$$

$$R_{12}^{+} - \frac{1}{\mathrm{Re}_{\tau}} \frac{\mathrm{d}U^{+}}{\mathrm{d}\tilde{y}} = \tilde{y} - 1 \qquad \tilde{y} = \frac{y}{H}, \quad y^{+} = \frac{y}{l^{+}} = \mathrm{Re}_{\tau}\, \tilde{y}$$

- For $\mathrm{Re}_{\tau} \to \infty$, for fixed y^{+} : $R_{12}^{+} - \partial U_i^{+} / \partial y^{+} = -1$

$$U^{+}(y) = f_i(y^{+}), \quad R_{12}^{+}(y) = R_i(y^{+})$$

- For $\mathrm{Re}_{\tau} \to \infty$, for fixed $\tilde{y}$: $R_{12}^{+} = \tilde{y} - 1$

$$R_{12}^{+}(y) = R_o(\tilde{y}) \Rightarrow \quad G_o(\tilde{y}, \partial U^{+} / \partial \tilde{y}) = R_o(\tilde{y})$$

$$\Rightarrow \partial U^{+} / \partial \tilde{y} = F_o(\tilde{y}) \Rightarrow \quad U_c^{+} - U^{+}(y) = f_o(\tilde{y}) \qquad (U_c = U(H))$$

"Outer" Equation

- The outer equation is:

$$R_{12}^{+} = \tilde{y} - 1$$

- Let's use an eddy viscosity model and integrate from the channel center:

$$R_{12} = -\nu_T(y,U)\frac{\partial U}{\partial y}$$

$$\nu_T = \kappa u_\tau y\left[1 - \frac{y}{2H}\right] \quad \text{(symmetric around centerline)}$$

$$R_{12}^{+} = \tilde{y} - 1 \Rightarrow \nu_T \frac{\partial U}{\partial y} = u_\tau^2\left[1 - \frac{y}{H}\right]$$

“Outer” Equation

- Integrating from channel center

$$\frac{\partial U^+}{\partial \tilde{y}} = F(\tilde{y}) \Rightarrow U_c^+ - U^+(y) = G(\tilde{y})$$

- In this particular case..

$$U^+(y) = U_c^+ + \frac{1}{\kappa}\log[\tilde{y}(2-\tilde{y})]$$

  - But we cannot use this to calculate τ_w
  - We need to connect this velocity profile to “inner” layer velocity profile

Log-Law From Matched Asymptotics (Pope, Section 7.1)

- Outer region: define velocity defect function

$$U_c^+ - U^+ = f_o(\tilde{y}) \text{ where } U_c = U(H)$$

- Inner region:

$$U^+ = f_i(y^+)$$

- At $\mathrm{Re}_\tau \to \infty$

$$\lim_{\tilde{y}\to 0} - \frac{\mathrm{d}f_o(\tilde{y})}{\mathrm{d}y} = \lim_{y^+\to\infty} \frac{\mathrm{d}f_i(y^+)}{\mathrm{d}y} \Rightarrow -\frac{1}{H} f'_0(\tilde{y}) = \frac{u_\tau}{\nu} f'_i(y^+)$$

$$\Rightarrow \boxed{-\tilde{y} f'_0(\tilde{y}) = y^+ f'_i(y^+) = \frac{1}{\kappa}}$$

Log-law From Matched Asymptotics

- Outer solution:

For $\tilde{y} \to 0$ $$U_c^+ - U^+ = f_o(\tilde{y}) = -\frac{1}{\kappa}\log\tilde{y} + B_0$$

- Inner Solution:

For $y^+ \to \infty$ $$U^+ = f_i(y^+) = \frac{1}{\kappa}\log y^+ + B_i$$

- Add eqns above (in the log-law region)
  - Centerline velocity is Re dependent

$$U_c^+ = \frac{1}{\kappa}\log \mathrm{Re}_\tau + \left[B_i + B_0\right] \text{ or } \lim_{\mathrm{Re}_\tau \to \infty} \frac{U_c}{u_\tau} = \infty$$

Going back to our example..

- Exact solution in outer layer is:

$$U_c^+ - U^+(y) = -\frac{1}{\kappa}\log[\tilde{y}(2-\tilde{y})] = f_o(\tilde{y})$$

$$\lim_{\tilde{y}\to 0} f_o(\tilde{y}) = -\frac{1}{\kappa}\log\tilde{y} - \frac{1}{\kappa}\log 2$$

$$\Rightarrow B_O = -\frac{1}{\kappa}\log 2$$

$$U_c^+ = \frac{1}{\kappa}\log\frac{\mathrm{Re}_\tau}{2} + B_i$$

$$\text{or..}\quad \frac{U_c}{u_\tau} = \frac{1}{\kappa}\log\frac{u_\tau \nu}{2H} + B_i$$

We can find wall shear stress from centerline velocity !

Log-Law in Channel Flow

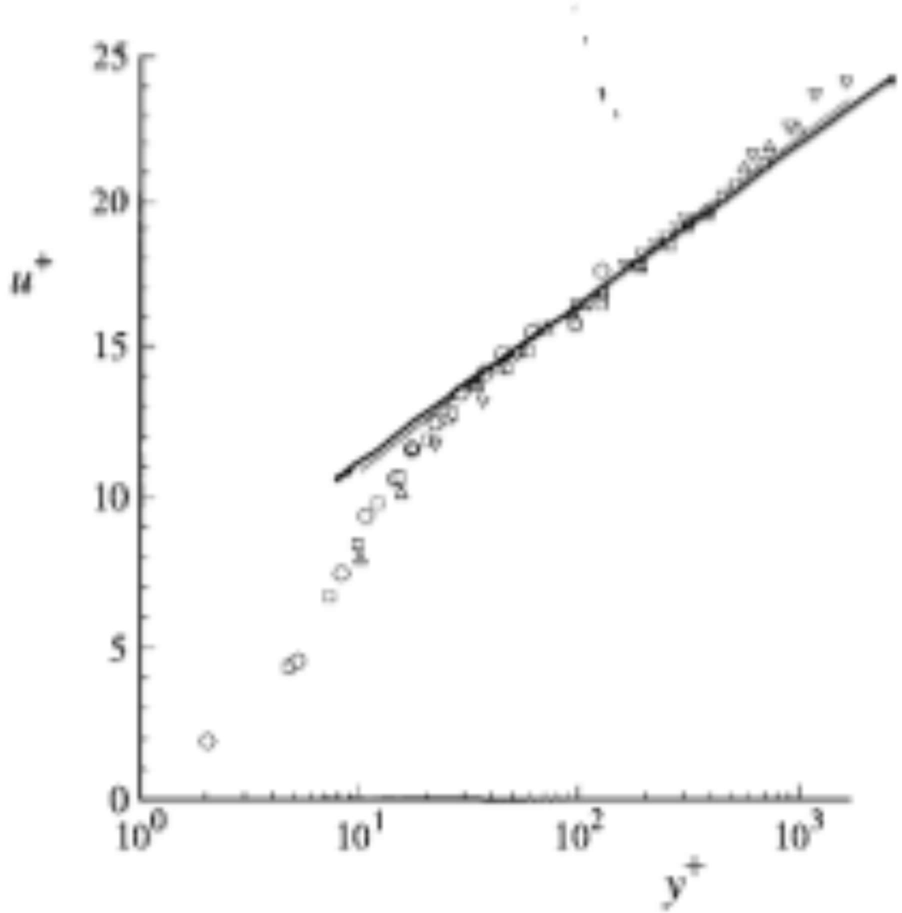


Fig. 7.7. Mean velocity profiles in fully developed turbulent channel flow measured by Wei and Willmarth (1989): ○, $Re_0 = 2{,}970$; □, $Re_0 = 14{,}914$; Δ, $Re_0 = 22{,}776$; ∇, $Re_0 = 39{,}582$; line, the log law, Eqs. (7.43)–(7.44).

Defect Law

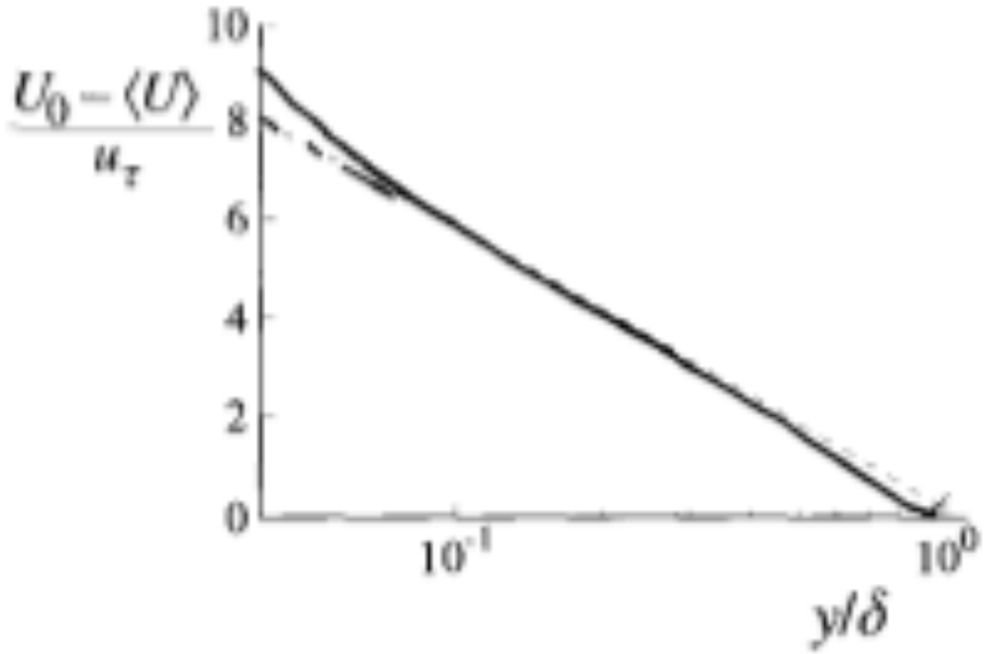
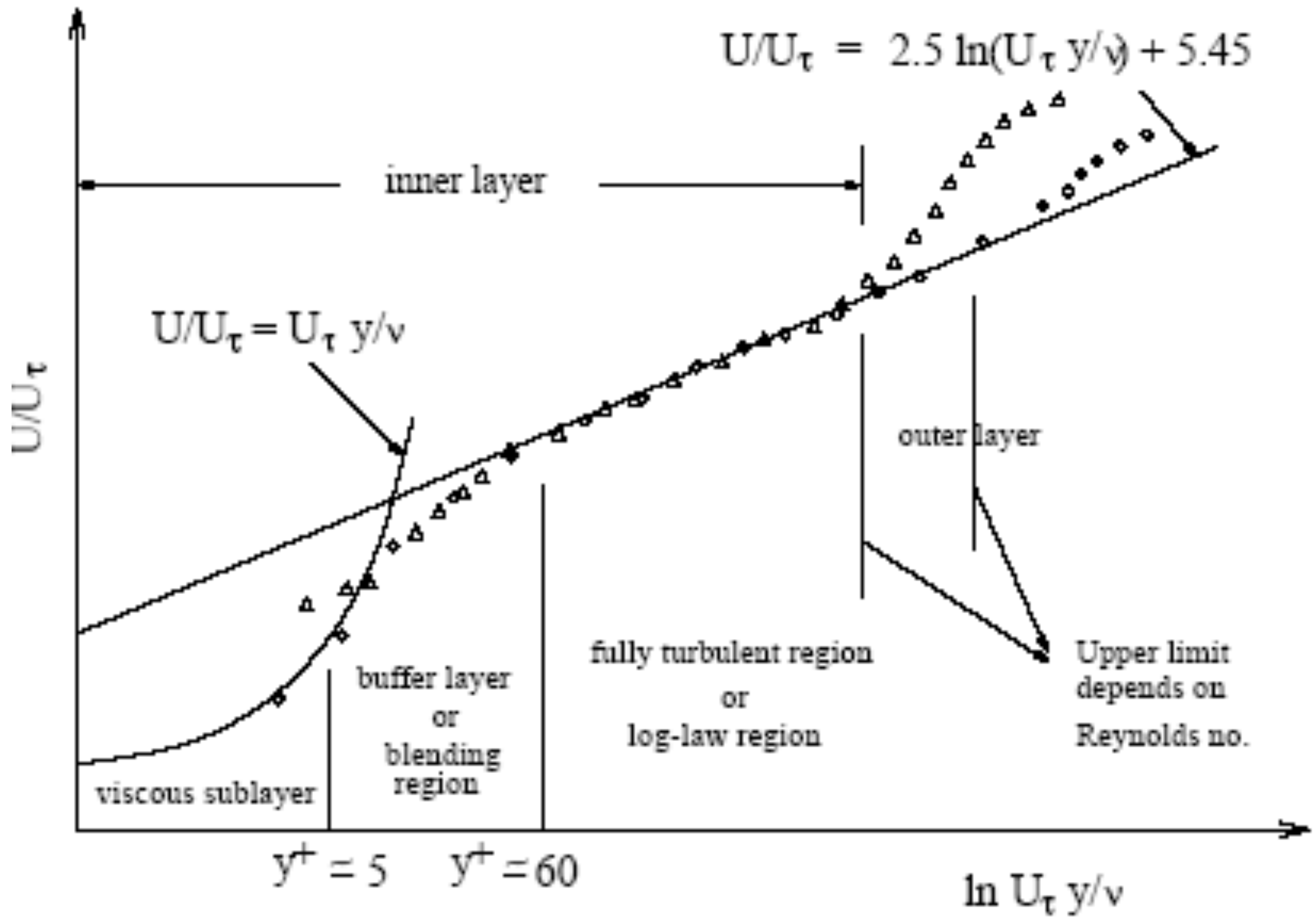


Fig. 7.9. The mean velocity defect in turbulent channel flow. Solid line, DNS of Kim *et al.* (1987), Re = 13,750; dashed line, log law, Eqs. (7.43)–(7.44).

Different Regions in Channel Flow Profile



Implications of Matched Asymptotics (Outer scaling)

- For $\tilde{y} \to 0$ the following *has* to be true:

$$\frac{\partial U^+}{\partial \tilde{y}} = -f'_0(\tilde{y}) = \frac{1}{\kappa \tilde{y}}$$

  - Implications for eddy viscosity models

$$\lim_{\tilde{y}\to 0} R_{12}^+(y) = \lim_{\tilde{y}\to 0} \nu_T(\tilde{y}) \frac{\partial U^+}{\partial \tilde{y}} = u_\tau^2 \Rightarrow \lim_{\tilde{y}\to 0} \nu_T(\tilde{y}) = \frac{\tilde{y} u_\tau^2}{\kappa}$$

  - Log law holds *only* in "log-law region"

In general, $U_c^+ - U^+ = f_o(\tilde{y})$ can be any function for $\tilde{y} \sim O(1)$

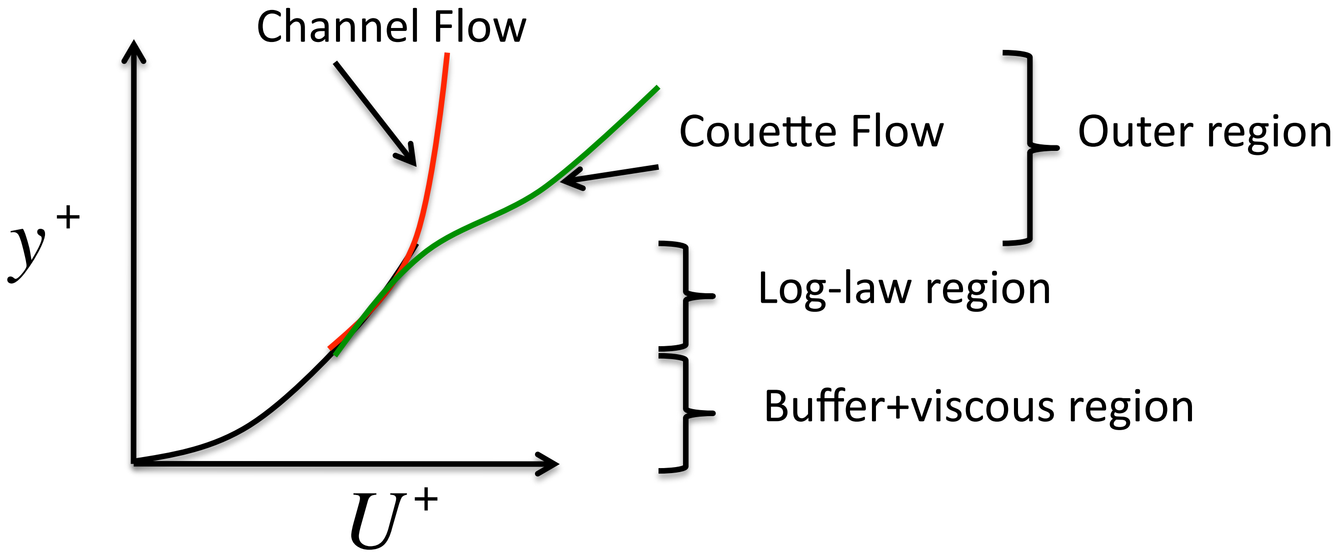
$$\lim_{\tilde{y}\to 0} \left[U_c^+ - U^+ \right] = -\frac{1}{\kappa} \log \tilde{y} + B_0 \quad \text{(log - law region)}$$

Implications of Matched Asymptotics (Inner Scaling)

- For $y^+ \to \infty$ the following *has* to be true:

$$U^+ = f_i(y^+) = \frac{1}{\kappa}\log y^+ + B_i$$

  - B_i=5.5 will *not* depend on outer flow (Universal constant)



Friction Factor From Matched Asymptotics (Channel Flow)

- Relation between center-line velocity and wall-shear stress:

$$\frac{U_c}{u_\tau} = \frac{1}{\kappa}\log\frac{u_\tau \nu}{2H} + B_i$$

- Friction factor:

$$C_f = \frac{\tau_w}{U_c^2/2} = \frac{2}{\left(U_c^+\right)^2} \qquad\qquad C_f = \frac{18}{\mathrm{Re}_b} \text{ for laminar flow}$$

$$\sqrt{\frac{2}{C_f}} = \frac{1}{\kappa}\log\left(\mathrm{Re}_b\sqrt{\frac{C_f}{2}}\right) + B_i - \frac{1}{\kappa} \text{ for turbulent flow}$$

C_f vs Re

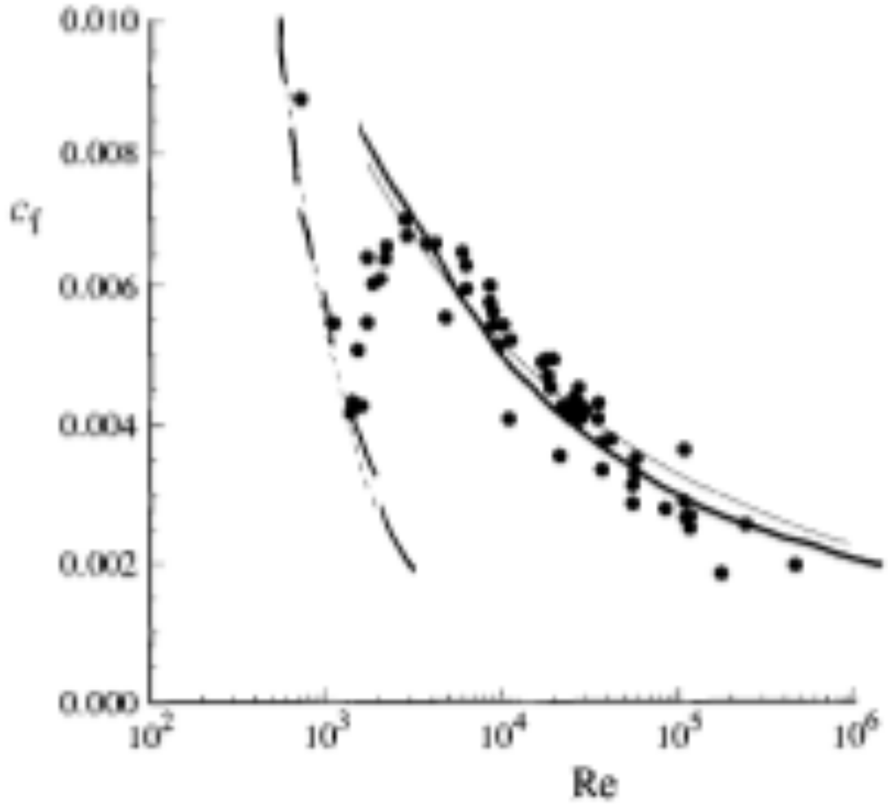
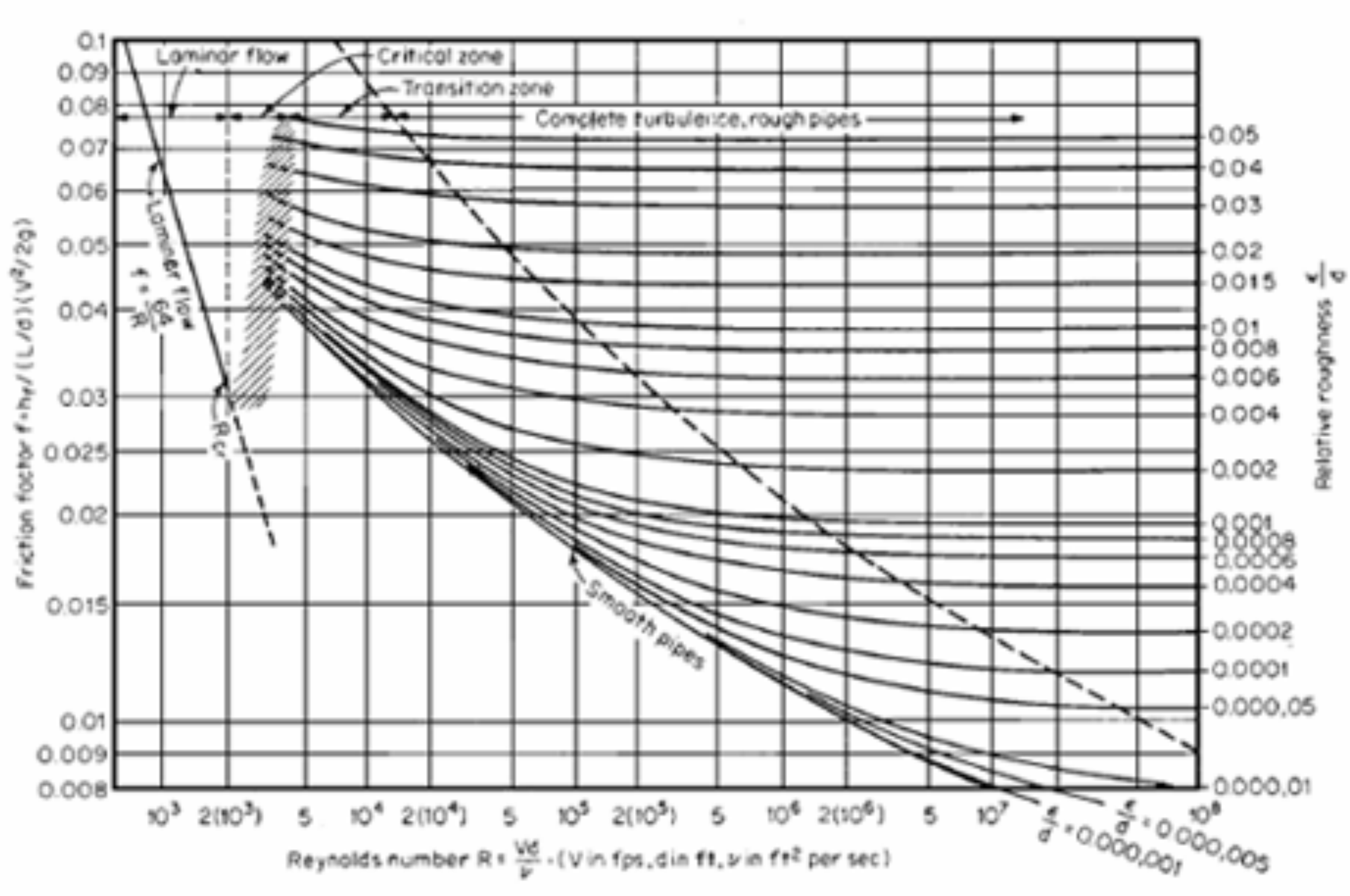


Fig. 7.10. The skin-friction coefficient $c_f = \tau_w / (\frac{1}{2}\rho U_0^2)$ against the Reynolds number ($\mathrm{Re} = 2\bar{U}\delta/\nu$) for channel flow: symbols, experimental data compiled by Dean (1978); solid line, from Eq. (7.55); dashed line, laminar friction law, $c_f = 16/(3\mathrm{Re})$.

Friction factor for pipe



Log-law From Mixing Length Model

- Let us look at $y^+ >> 1, \quad \tilde{y} << 1$

$$R_{12} \approx -u_\tau^2$$

- Mixing length model:

Karman Constant

$$R_{12} = -l_{mix}^2 \left(\partial U / \partial y\right)^2; \qquad l_{mix} = \kappa y$$

$$\left(\kappa y\right)^2 \left(\partial U / \partial y\right)^2 = u_\tau^2 \qquad \Rightarrow \qquad \kappa y \, \partial U / \partial y = u_\tau$$

$$U = \frac{u_\tau}{\kappa} \log y + \text{Constant}$$

- Cannot integrate upto wall
  - Re dependence is not clear

Log-law from mixing length model

- Stress balance near the wall:

$$-R_{12} + \nu \frac{\mathrm{d}U}{\mathrm{d}y} = \tau_w \quad (\text{for } \tilde{y} << 1)$$

$$\Rightarrow l_{mix}^2(y^+)\left(\frac{\mathrm{d}U^+}{\mathrm{d}y^+}\right)^2 + \frac{\mathrm{d}U^+}{\mathrm{d}y^+} = 1 \quad (\text{From mixing length model})$$

- Solve quadratic eqn for $\partial U^+ / \partial y^+$ and integrate

$$U^+(y) = \int_0^{y^+} \frac{2dy'}{1 + [1 + 4l_{mix}^+(y')^2]^{1/2}}$$

Van Driest Damping Factor

- Reynolds stress ~ y^3 near the wall

$$u' \sim O(y), \quad w' = O(y), \quad v' = O(y^2) \quad \text{(From continuity)}$$

$$\Rightarrow \overline{u'v'} = O(y^3)$$

- But $l_{\mathrm{mix}}(y) = \kappa y \Rightarrow \overline{u'v'} = \kappa^2 y^2 \left[\partial U / \partial y \big|_{y=0} \right]^2 = O(y^2)$
- In viscous layer, mixing length needs to be 'damped' (Van Driest 1956):

$$l_{\mathrm{mix}}(y) = \kappa y \left[1 - \exp(-y^+ / A_0^+) \right] = O(y^2)$$

$$A_0^+ = 26$$

  - A_0^+ determines B_i

Van Driest Damping
Mixing length and eddy viscosity

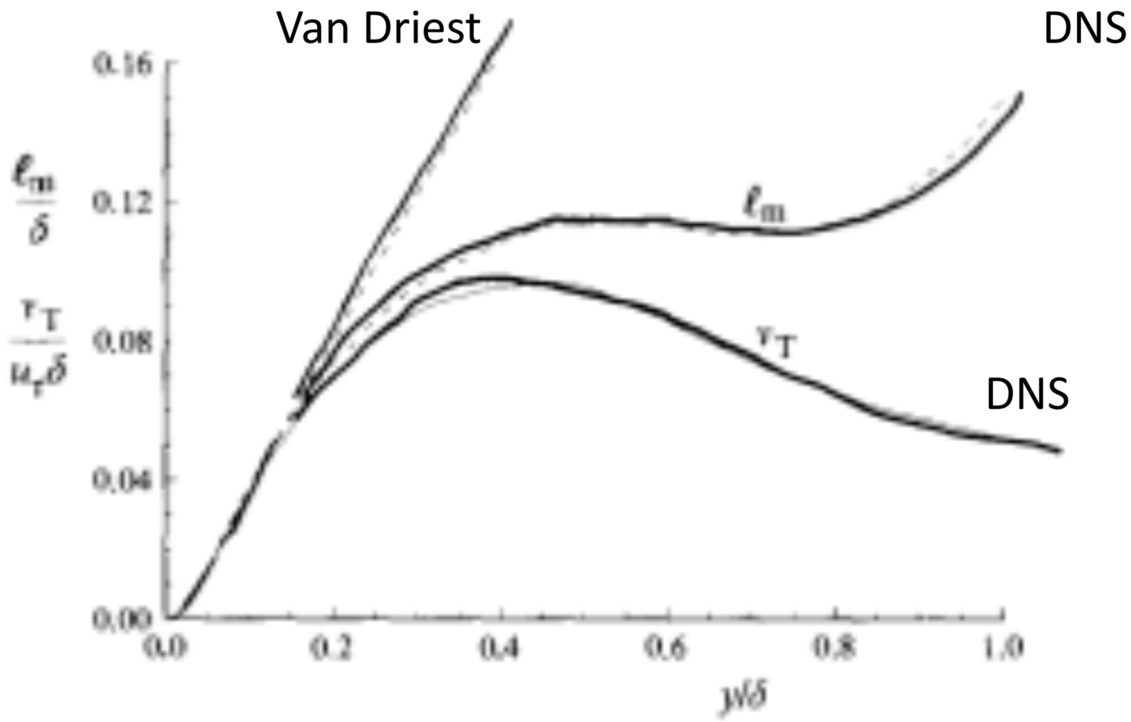


Fig. 7.30. Turbulent viscosity and mixing length deduced from direct numerical simulations of a turbulent boundary layer (Spalart 1988). Solid line, ν_T from DNS; dot–dashed line, ℓ_m from DNS; dashed line ℓ_m and ν_T according to van Driest's specification (Eq. (7.145)).

Van Driest Damping Velocity Profile

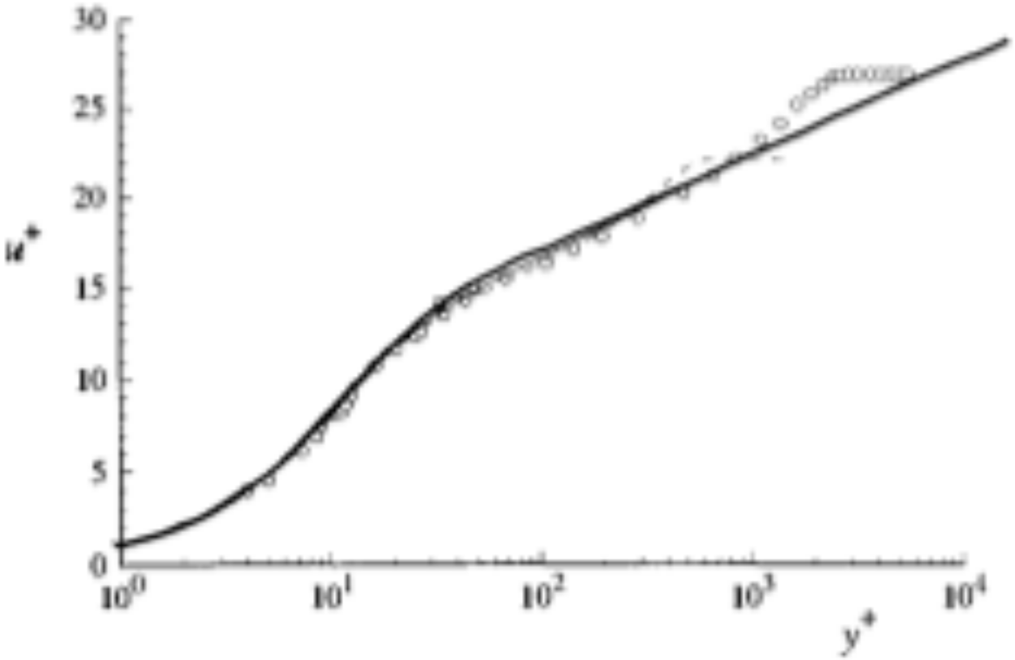


Fig. 7.27. Mean velocity profiles in wall units. Circles, boundary-layer experiments of Klebanoff (1954), $Re_\theta = 8{,}000$; dashed line, boundary-layer DNS of Spalart (1988), $Re_\theta = 1{,}410$; dot–dashed line, channel flow DNS of Kim *et al.* (1987), $Re = 13{,}750$; solid line, van Driest's law of the wall, Eqs. (7.144)–(7.145).