11. Instability

Drazin & Reid (1981) define hydrodynamic (in)stability as that branch of hydrodynamics concerned with "when and how laminar flows break down, their subsequent development, and their eventual transition to turbulence." From this definition, we can propose the following general procedure for studying hydrodynamic stability mathematically:

1. Start with a laminar solution to the hydrodynamic equations (e.g. Mass conservation and Navier-Stokes equations).
2. Perturb the laminar solution with small disturbances, usually sinusoidal in space and time.
3. Substitute the disturbed solution into the hydrodynamic equations to derive disturbance equations. This generally yields an eigenvalue problem for the wave number and frequency of the disturbances.
4. Solve the eigenvalue problem to study instability: solutions of the perturbed equations for which the disturbances grow in time are called *absolute instabilities* and solutions which grow in space are called *convective instabilities*; solutions that are damped are *stable*.

In this chapter we illustrate stability analysis through two classic examples: Kelvin-Helmholtz instability and general shear-flow instability. For Kelvin-Helmholtz instability, we first use a heuristic approach to derive the Richardson number and to give a physical understanding (intuition) of the flow and instability mechanism. We then implement a classical stability analysis (following the steps outlined above) to compliment the heuristic approach. For the general shear flow instability, we again use our general analysis approach, and derive the classic Rayleigh inflection-point theorem, a necessary, but not sufficient, condition for inviscid shear flow instability.

For further reading on hydrodynamic instability consult, among others, Drazin (2002), Drazin & Reid (1981), Acheson (1990), Joseph (1976), Iooss (1990), Lin (1966), Betchov & Criminale (1967), Chandrasekhar (1961) and a wealth of journal articles, especially in the *Journal of Fluid Mechanics*.

11.1 Kelvin-Helmholtz instability

Kelvin-Helmholtz (K-H) instability, shown experimentally in Figure 11.1, is a classic type of instability generated in density stratified shear flows. The instability develops when small waves at the picnocline (region of steepest density gradient) become unstable and begin to role up into the characteristic K-H billows. The K-H instability enhances mixing both mechanically

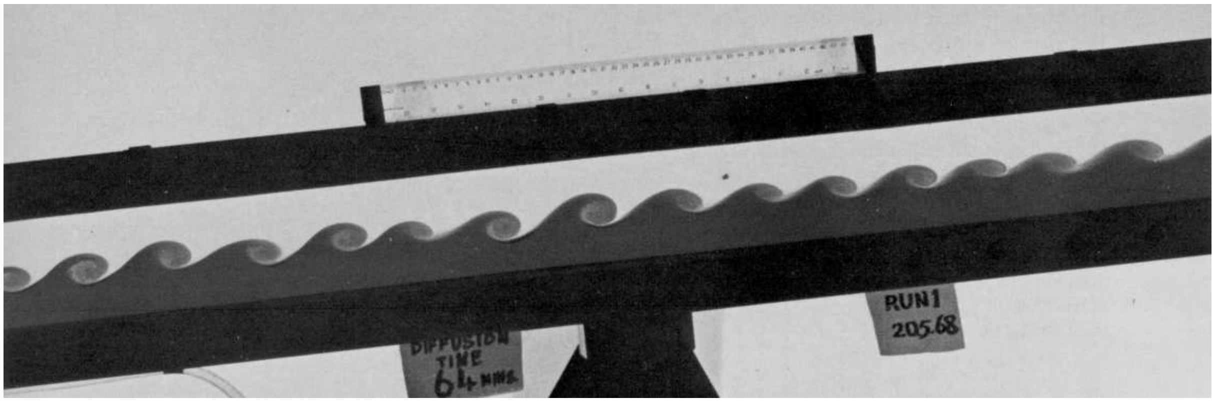

Fig. 11.1. Kelvin-Helmholtz instability of stratified shear flow taken from van Dyke (1982).

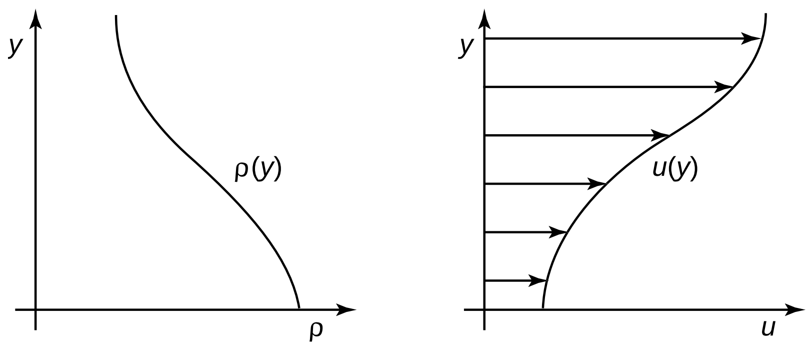

Fig. 11.2. Density stratified shear flow profiles.

(through roll-up of the billows) and diffusively (through the creation of strong density gradients and mixing within the billows themselves).

In this section we seek to predict the conditions necessary for the generation of K-H instability: first through a heuristic argument, and then later through a more formal inviscid theoretical analysis.

11.1.1 Heuristic approach

Consider the density and velocity fields depicted in Figure 11.2. To test the stability of such a stratified shear flow, imagine the behavior of a packet of fluid initially at y that moves up to $y + \eta$, where η is a small vertical displacement (refer to Figure 11.3). To satisfy continuity, another fluid packet must also move from $y+\eta$ to y. Due to the stratification, the fluid packet at the new level ($y+\eta$ or y) experiences a buoyant force directed back to the original position. Due to the shear flow, however, kinetic energy (KE) is advected with the fluid packet, and the packet is accelerated as it joins the new fluid layer. As a rule-of-thumb, *instability can be expected when the KE lost by moving into another layer is greater than the buoyant work required to move the fluid packet to the new layer.*

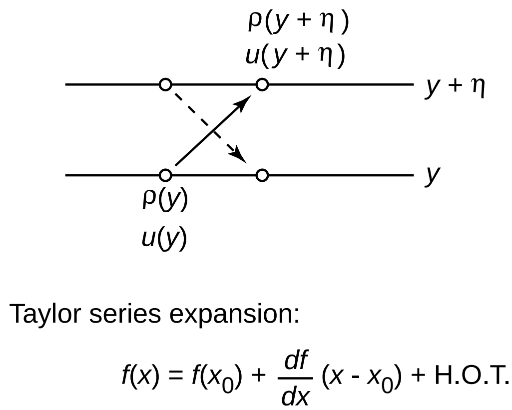

Fig. 11.3. Small displacement in density stratified shear flow.

To calculate the buoyant work needed to move the heavy fluid (at the point y), consider the buoyant force acting on the fluid packet. The change in buoyancy, $\varDelta B$, of the fluid packet moving from y to $y+\eta$ is given by

$$
\begin{aligned}
\varDelta B &= g\rho(y) - g\rho(y+\eta) \\
&= g\rho(y) - g\left(\rho(y) + \frac{d\rho}{dy}\eta + ...\right) \\
&= -g\frac{d\rho}{dy}\eta
\end{aligned}
\tag{11.1}
$$

where we use a first-order Taylor series expansion in line two to calculate $\rho(y+\eta)$ (refer to Figure 11.3). The work done on the fluid packet is the buoyancy integrated along the path, δy, giving

$$
\int_0^{\delta y} -g\frac{d\rho}{dy}\eta d\eta = -g\frac{d\rho}{dy}\frac{(\delta y)^2}{2}. \tag{11.2}
$$

Since the change in buoyancy is the same for the lighter fluid moving down, the total work is the sum of both contributions, giving

$$
W_B = -g\frac{d\rho}{dy}(\delta y)^2. \tag{11.3}
$$

To calculate the change in KE, first consider the total KE before the fluid packets exchange place. Using the Boussinesq approximation, the density gradient is ignored, and the KE is calculated from the mean density ρ_0 giving

$$
\begin{aligned}
\mathrm{KE}_1 &= \frac{\rho_0}{2}\left(u^2 + (u+\delta u)^2\right) \\
&= \frac{\rho_0}{2}\left(2u^2 + 2u\delta u + (\delta u)^2\right)
\end{aligned}
\tag{11.4}
$$

where u is the velocity at the level y and δu is the velocity at $y+\eta$. Once the fluid packets exchange place, we can calculate the new KE by assuming that each particle takes on the mean velocity between the two levels:

$$\begin{aligned}
\mathrm{KE}_2 &= \frac{\rho_0}{2}\left(2\left(\frac{u+(u+\delta u)}{2}\right)^2\right) \\
&= \frac{\rho_0}{8}\left(8u^2 + 8u\delta u + 2(\delta u)^2\right) \\
&= \frac{\rho_0}{2}\left(2u^2 + 2u\delta u + \frac{1}{2}(\delta u)^2\right).
\end{aligned} \tag{11.5}$$

The total change in KE is given by

$$\varDelta\mathrm{KE} = -\frac{\rho_0}{4}(\delta u)^2, \tag{11.6}$$

and this is the amount of kinetic energy lost by exchanging the two fluid packets.

Using our heuristic definition for when to expect instability, we have

$$\begin{aligned}
\varDelta\mathrm{KE} &> W_B \\
\frac{\rho_0}{4}(\delta u)^2 &> -g\frac{d\rho}{dy}(\delta y)^2 \\
\frac{1}{4} &> -\frac{g}{\rho}\frac{d\rho}{dy} \bigg/ \left(\frac{du}{dy}\right)^2 .
\end{aligned} \tag{11.7}$$

We recognize the right-hand side of (11.7) from Chapter 8 as the gradient Richardson number (Ri). As our heuristic derivation clearly points out, Ri is the ratio of the work required for mixing done against buoyancy to the kinetic energy available from to the shear velocity profile. The Richardson number is an important parameter used to describe the stability of stratified shear flows. The threshold of 1/4 is often used to predict the onset of reservoir mixing (usually due to seiching or wind mixing), in particular thermocline deepening. Here, it provides a rule-of-thumb for the onset of Kelvin-Helmholtz instability in a density stratified shear flow.

11.1.2 Theoretical approach

We now follow the mathematical procedure outlined in the introduction to this chapter. As we will see, the main challenge in a theoretical stability analysis is to find the appropriate equations and boundary conditions.

Consider the two-layer system depicted in Figure 11.4. For our analysis we will make the following simplifying assumptions:

1. The two fluids are immiscible.
2. Inviscid analysis is applicable (i.e. the viscosity approaches zero, implying the Reynolds number, Re, is large).
3. The flow is irrotational. This assumption follows Kelvin's original analysis and is valid if the initial disturbances are irrotational since irrotational flow persists in inviscid theory. If we extend the analysis to include rotational flow, however, the criterion for instability turns out to be no more restrictive; thus, Kelvin's analysis provides a necessary and sufficient stability criterion (Drazin 2002).
4. Convective terms of the momentum equation and boundary conditions can be linearized (implying the disturbances are small compared to the base flow).

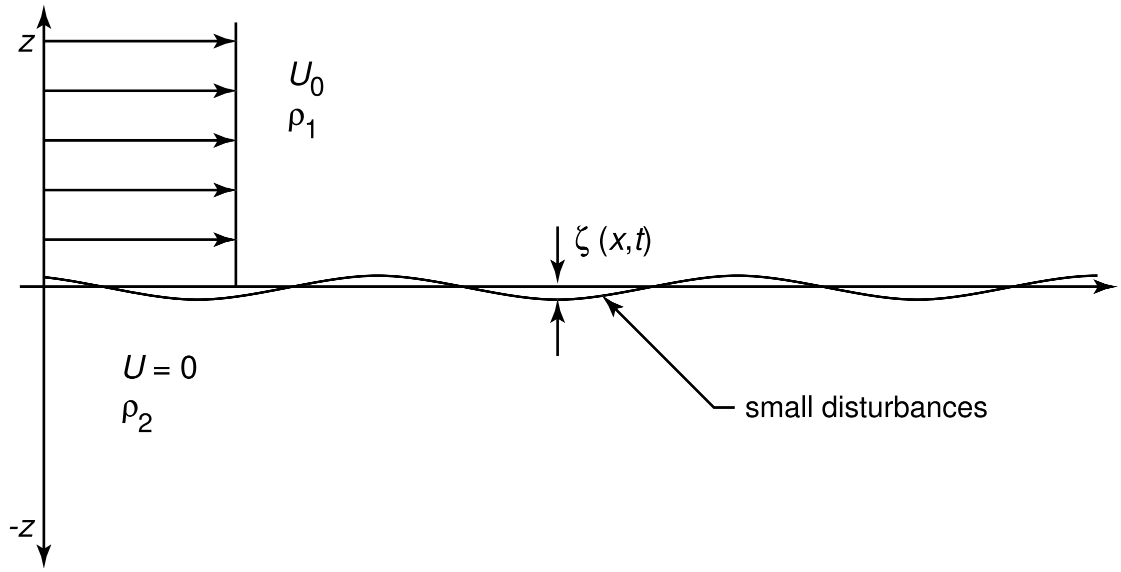

Fig. 11.4. Definition sketch for K-H theoretical stability analysis.

To applying the following results to a natural system, these assumptions must be verified. Since these are common assumptions, we expect this analysis to have wide application to environmental flows.

The first step of our general approach is two define the basic laminar flow conditions. From the profiles in Figure 11.4, we have for the upper layer

$$\boldsymbol{u} = U_0 \boldsymbol{\imath} \tag{11.8}$$

$$\rho(z) = \rho_1 \tag{11.9}$$

$$p(z) = p_0 - g\rho_1 z \tag{11.10}$$

and for the bottom layer

$$\boldsymbol{u} = 0 \tag{11.11}$$

$$\rho(z) = \rho_2 \tag{11.12}$$

$$p(z) = p_0 - g\rho_2 z \tag{11.13}$$

where U_0 is the uniform velocity in the upper layer, p_0 is the pressure at the interface, ρ_1 and ρ_2 are the densities of the upper and lower fluids, respectively, and z is positive upward and zero at the interface. These equations are, thus, one possible laminar inviscid solution to a density shear flow.

The second step is to impose small disturbances to the base laminar flow. Without yet specifying their form, we define the velocity disturbances in the upper and lower layers as $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, respectively. Thus, the disturbed velocity profiles become

$$\boldsymbol{u}_1 = U_0 \boldsymbol{\imath} + \boldsymbol{q}_1 \tag{11.14}$$

$$\boldsymbol{u}_2 = \boldsymbol{q}_2. \tag{11.15}$$

Substituting the disturbances into the conservation of mass and the inviscid Navier-Stokes equations and linearizing the convective terms, we obtain the following set of governing equations:

$$\boldsymbol{\nabla} \cdot \boldsymbol{q}_1 = \boldsymbol{\nabla} \cdot \boldsymbol{q}_2 = 0 \tag{11.16}$$

$$\frac{\partial \boldsymbol{q}_1}{\partial t} + U_0 \frac{\partial \boldsymbol{q}_1}{\partial x} = \frac{\boldsymbol{\nabla} p_1}{\rho_1} \tag{11.17}$$

$$\frac{\partial \boldsymbol{q}_2}{\partial t} = \frac{\boldsymbol{\nabla} p_2}{\rho_2} \tag{11.18}$$

where p_1 and p_2 are the dynamic pressures of the disturbances. To complete step two, however, we must still specify the boundary conditions.

We first specify a kinematic boundary condition: fluid particles can only move tangentially to the fluid interface. The interface location is defined by the function

$$F = z_i - \zeta(x,t) = 0 \tag{11.19}$$

where $\zeta(x,t)$ is the interface disturbance (refer to Figure 11.4). The normal velocity, $\boldsymbol{q}_s$, at the interface is given by the material derivative of F:

$$\frac{\partial F}{\partial t} + \boldsymbol{q}_s \cdot \nabla F = 0. \tag{11.20}$$

After linearization, we obtain for the top and bottom layers, respectively,

$$\frac{\partial \zeta}{\partial t} + U_0 \frac{\partial \zeta}{\partial x} = w_1 \tag{11.21}$$

$$\frac{\partial \zeta}{\partial t} = w_2 \tag{11.22}$$

taken at $z = 0$, where w_1 is the vertical velocity in the upper layer and w_2 is the vertical velocity in the lower layer.

The second boundary condition is a dynamic one: the normal stress of the fluid is continuous at the interface (Drazin 2002). For an inviscid fluid, this means that the pressure is continuous at the interface. For irrotational flow, the total pressure at the interface has a dynamic and gravitational component, giving

$$p_1 - \rho_1 g \zeta = p_2 - \rho_2 g \zeta \tag{11.23}$$

applied at the linearized interface, $z = 0$.

The final boundary condition is that the disturbances die away far from the interface (at $z = \pm\infty$).

The third step in our analysis is to substitute the disturbances into the governing equations. At this point we must specify the form of the disturbances. Assuming sinusoidal disturbances and applying separation of variables, we have for the upper layer

$$\begin{pmatrix} p_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} P_1 \\ W_1 \end{pmatrix} e^{-kz} e^{i(kx - \omega t)}, \tag{11.24}$$

for the lower layer

$$\begin{pmatrix} p_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} P_2 \\ W_2 \end{pmatrix} e^{kz} e^{i(kx - \omega t)} \tag{11.25}$$

and for the interface

$$\zeta = Ze^{i(kx-\omega t)} \tag{11.26}$$

where k and ω are the wave number and wave frequency of the disturbances, the capitol letters are constant coefficients and the sinusoidal functions are written with implied real-part operators (i.e. we only retain $\mathrm{Re}(Ae^{i\omega t})$). Note that the chosen z-dependence in the disturbance equations already satisfies the final boundary condition. Because the flow is inviscid, these disturbances also form a velocity potential ϕ, and the conservation of mass equations (11.16) are satisfied implicitly.

Now we can define when the flow becomes unstable. If we assume the wave number, k, is real, and allow ω to be complex, we can rearrange the exponential in the above equations to give:

$$e^{-i\omega t} = e^{-i(\omega_r + i\omega_i)t} = e^{-i\omega_r t}e^{\omega_i t}. \tag{11.27}$$

The first term, $e^{-i\omega_r t}$, is just a sinusoidal function. The second term, $e^{\omega_i t}$, is a monotonic function of time. If ω_i is less than zero, this term becomes a damping function and we have stability; if ω_i is equal to zero, we have neutral stability (the disturbances neither grow nor decline); and, if ω is greater than zero, the final term becomes a growth function and we have instability.

Substituting the disturbances (11.24) and (11.25) into the remaining governing equations (11.17) and (11.18) and boundary conditions (11.21), (11.22), and (11.23), we obtain the following matrix equation

$$\begin{pmatrix} i(kU_0-\omega) & -1 & 0 & 0 & 0 \\ -i\omega & 0 & -1 & 0 & 0 \\ g(\rho_2-\rho_1) & 0 & 0 & 1 & -1 \\ 0 & 0 & -i\omega & 0 & k/\rho_2 \\ 0 & i(kU_0-\omega) & 0 & -k/\rho_1 & 0 \end{pmatrix} \begin{pmatrix} Z \\ W_1 \\ W_2 \\ P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{11.28}$$

which is a classical Eigenvalue problem for ω and k. To have a non-trivial solution the determinant of the coefficient matrix must be zero, which leads to

$$(\rho_1+\rho_2)\omega^2 - 2kU_0\rho_1\omega + k^2U_0^2\rho_1 - kg(\rho_2-\rho_1) = 0. \tag{11.29}$$

This equation specifies the relationship between k and ω and is called the *dispersion relation*. Applying the quadratic equation, we can obtain solutions for ω:

$$\omega = \frac{kU_0\rho_1 \pm i\sqrt{k^2U_0^2\rho_1\rho_2 - kg(\rho_2-\rho_1)(\rho_1+\rho_2)}}{(\rho_1+\rho_2)} \tag{11.30}$$

thus, we now have a general criteria to judge the likelihood of instability.

11.1.3 Applications of the results

To investigate the behavior of (11.30), consider the following special cases (adapted from Drazin (2002)). As we see in this section, the work of doing a theoretical analysis pays off by providing information to a wide range of applications.

Surface gravity waves. First, consider the case $U_0 = \rho_1 = 0$, which is the model of surface gravity waves in deep water. (11.30) then reduces to

$$\omega = \pm\sqrt{kg} \tag{11.31}$$

which doesn't have an imaginary part and is, thus, always stable. Calculating the wave speed $c = \omega/k$, we obtain the well know result for linear deep water waves

$$c = \sqrt{\frac{g}{k}}. \tag{11.32}$$

Since our linear analysis shows that surface gravity waves are absolutely stable, we would expect that non-linear effects are responsible for surface wave breaking. This turns out to be true: non-linear effects create wave steepening which leads to wave breaking.

Internal gravity waves. Now consider the case of internal gravity waves, where $U_0 = 0$ and $\rho_1 > 0$. In this case the dispersion relation gives

$$w = \frac{\pm\sqrt{kg(\rho_2^2 - \rho_1^2)}}{(\rho_1 + \rho_2)} \tag{11.33}$$

which is stable only for $\rho_2 > \rho_1$; if the heavier fluid is on top, we have instability. The stable wave speed is

$$c = \pm\sqrt{\frac{g(\rho_2 - \rho_1)}{k(\rho_1 + \rho_2)}} \tag{11.34}$$

indicating that the internal wave phase speed is *reduced* from the surface gravity wave phase speed ($\sqrt{g/k}$) by the normalized density difference between the layers.

Kelvin-Helmholtz instability. So far we have shown that as long as the fluids are stably stratified and not moving the system remains stable. Now we consider the case of the upper fluid moving: $|U_0| > 0$. From the dispersion relation we now have the stability criterion

$$U_0^2 \leq \frac{g}{|k|\rho_1\rho_2}(\rho_2^2 - \rho_1^2); \tag{11.35}$$

otherwise, ω has imaginary parts and the flow is unstable. This equation shows that the density differences, through the effects of buoyancy, provide a means of stabilization. When the densities are equal, the right-hand-side is zero and for any U_0, the flow becomes unstable. As the density differences grow, the right-hand-side becomes larger and stronger shear flows remain stable.

Surface tension can also provide stabilization (see Acheson (1990)), and for surface tension T, the stability relation becomes

$$U_0^2 \leq \frac{(\rho_1 + \rho_2)}{\rho_1\rho_2}\left(|k|T + \frac{g}{|k|}(\rho_2 - \rho_1)\right). \tag{11.36}$$

We can also compare these theoretical results to those developed in our heuristic approach. To do this, we need to define a characteristic vertical length-scale, L. One possible choice is to select L such that the non-dimensional wave number for the most unstable mode is of order 1, that is

$$k^* = k_0 L = \mathcal{O}(1) \tag{11.37}$$

where k^* is the dimensionless wave number and k_0 is the wave number that provides the largest disturbance growth rate. Using this length scale and assuming continuous profiles of velocity and density gradient, we can use Taylor expansion to define

$$\Delta u = U_1 - U_2 = \frac{du}{dz} L \tag{11.38}$$

$$\Delta \rho \ = \rho_1 - \rho_2 \ = \frac{d\rho}{dz}. \tag{11.39}$$

Substituting these results into (11.35) we obtain

$$\frac{kL}{2} > -\frac{g}{\rho_0}\frac{d\rho}{dz} \Bigg/ \left(\frac{du}{dy}\right)^2 \tag{11.40}$$

where we have used the Boussinesq approximation to write

$$\rho_1 + \rho_2 = 2\rho_0. \tag{11.41}$$

Instability is triggered through the most unstable mode of the disturbances; thus, from our definition of L, we have the stability criterion

$$Ri < \frac{1}{2} \tag{11.42}$$

which is in good agreement with our heuristic result ($Ri < 1/4$) and which has been verified experimentally. Most texts on reservoir mixing (e.g. Fischer et al. (1979)), however, use the 1/4-threshold to predict the onset of mixing, which is also in agreement with field reservoir experiments. This difference in criteria (1/4 versus 1/2) arises from the fact that the K-H billows (which initiate instability) begin to form below the threshold of 1/2, but they break to cause mixing beginning at the threshold of 1/4.

11.2 Shear flow instability

As a second example of instability analysis, we consider the general shear flow depicted in Figure 11.5. To stay general, the mathematics become a little more abstract, but we still follow the same stability analysis procedure.

11.2.1 Governing equation

The first two analysis steps are simple: for a general two-dimensional shear flow the laminar solution is given by $\boldsymbol{u} = (U(y), 0, 0)$, and the perturbed velocity profile by $\boldsymbol{q} = (U(y) + u(x, y), v(x, y), 0)$.

For the third step, we start by substituting the disturbed profile $\boldsymbol{q}$ into the conservation of mass and Navier-Stokes equations to obtain the disturbance equations. For mass conservation we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{11.43}$$

Substituting into the momentum equations we obtain

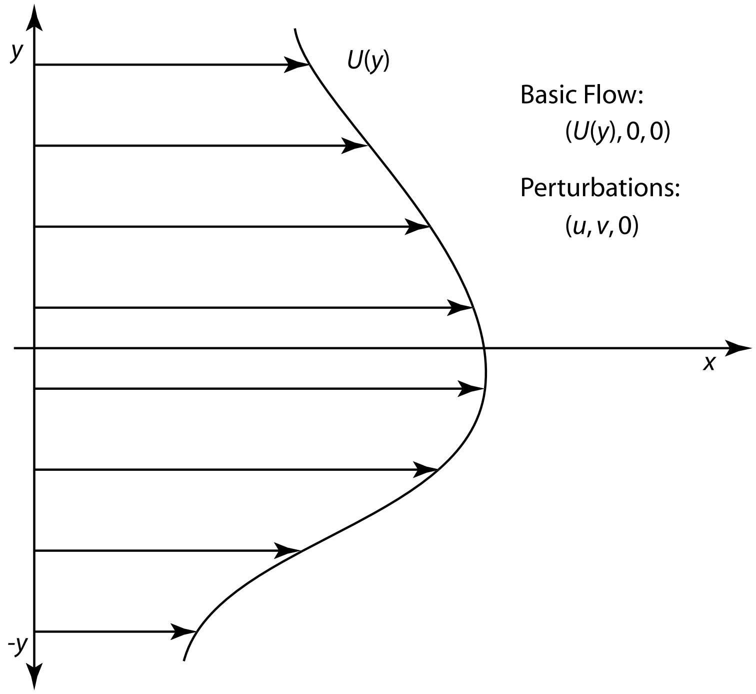

Fig. 11.5. General two-dimensional shear flow profile.

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{11.44}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{11.45}$$

were p is the dynamic pressure of the disturbances and were we have ignored quadratic terms of the disturbances (linearization). We can remove the dynamic pressure terms by taking the y-derivative of (11.44) and the x-derivative of (11.45) and subtracting. This gives the single momentum equation

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\partial^2 U}{\partial y^2}v + \frac{\partial U}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + U\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2}\right) \\ = \nu\left(\frac{\partial^3 u}{\partial x^2 \partial y} + \frac{\partial^3 u}{\partial y^3} - \frac{\partial^3 v}{\partial x^3} - \frac{\partial^3 v}{\partial x \partial y^2}\right).\end{aligned} \tag{11.46}$$

Since this flow is incompressible and two-dimensional, we simplify further by introducing the stream function, ψ, defined as

$$u = \frac{\partial \psi}{\partial y}; \;\; v = -\frac{\partial \psi}{\partial x}$$

which satisfies mass conservation (11.43) automatically. Substituting ψ into (11.46) gives the final form of our governing disturbance equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial x^2}\right)-\frac{\partial^2 U}{\partial y^2}\frac{\partial\psi}{\partial x}+U\left(\frac{\partial^3\psi}{\partial x\partial y^2}+\frac{\partial^3\psi}{\partial x^3}\right)=\nu\left(\frac{\partial^4\psi}{\partial x^4}+2\frac{\partial^4\psi}{\partial x^2\partial y^2}\frac{\partial^4\psi}{\partial y^4}\right). \quad (11.47)$$

We now define the disturbances in terms of the stream function. Assuming sinusoidal disturbances and using separation of variables we have

$$\psi(x,y,t)=f(y)e^{i(kx-\omega t)}. \quad (11.48)$$

Substituting into (11.47) and simplifying we obtain a new equation for f:

$$(U-c)f''+(k^2c-k^2U-U'')f=\frac{\nu}{ik}(k^4f-2k^2f''+f'''') \quad (11.49)$$

where c is the phase velocity, defined as $c=\omega/k$, and the prime-notation denotes derivatives with respect to y. The boundary conditions are that f and its first derivative vanish at the boundaries. Thus, as we expected, we now have an eigenvalue problem for k and ω (or c).

Before we leave step three, we non-dimensionalize (11.49) to make it more general. The non-dimensional variables (denoted by superscript $*$)are defined as

$$\begin{aligned} U &= U_0u^*;\ \ c=U_0c^* \\ f &= U_0Lf^*;\ \ k=(1/L)k^* \\ y &= Ly^*. \end{aligned}$$

Where U_0 and L are characteristic velocity and length scales of the shear flow profile. Substituting these variables and dropping the superscript notation, the non-dimensional governing equation becomes

$$(u-c)(f''-k^2f)-u''f=\frac{1}{ikR_e}(k^4f-2k^2f''+f'''') \quad (11.50)$$

where Re is the Reynolds number, defined as $Re=U_0L/\nu$. (11.50) is the celebrated Orr-Sommerfeld equation.

11.2.2 Rayleigh's inflection-point theorem

To derive the classic Rayleigh inflection-point theorem, we consider the inviscid Orr-Sommerfeld equation:

$$f''-k^2f-\frac{u''}{(u-c)}f=0 \quad (11.51)$$

with the boundary condition that f vanishes at the boundary (we can no longer enforce $f'=0$ at the boundaries since we are neglecting the fourth-order term).

For the flow to be unstable, ω must be complex, thus, c is complex. We then expect f to also be complex and we can write the following two equations:

$$f''-\left(k^2+\frac{u''}{(u-c)}\right)f=0 \quad (11.52)$$

$$f^{*\prime\prime}-\left(k^2+\frac{u''}{(u-c^*)}\right)f^*=0 \quad (11.53)$$

where $*$ now indicates the complex conjugate operator. Multiplying the first equation by f^* and the second by f and subtracting, we obtain

$$f''f^* - ff^{*\prime\prime} = \left(\frac{u''}{(u-c)} - \frac{u''}{(u-c^*)}\right) f^* f \tag{11.54}$$

To continue, we must recognize the identity

$$\begin{aligned} f''f^* - ff^{*\prime\prime} &= f^*f'' + f^{*\prime}f' - f^{*\prime}f' - ff^{*\prime\prime} \\ &= (f^*f' - f^{*\prime}f)' \end{aligned} \tag{11.55}$$

which lets us write

$$(f^*f' - f^{*\prime}f)' = \left(\frac{u''}{(u-c)} - \frac{u''}{(u-c^*)}\right) f^* f. \tag{11.56}$$

We can now integrate (arbitrarily taking the boundaries at $\pm\infty$) as follows

$$\begin{aligned} \int_{-\infty}^{\infty} (f^*f' - f^{*\prime}f)' dy &= \int_{-\infty}^{\infty} \left(\frac{u''}{(u-c)} - \frac{u''}{(u-c^*)}\right) f^* f dy \\ f^*f' - f^{*\prime}f|_{-\infty}^{\infty} &= \int_{-\infty}^{\infty} \left(\frac{u''}{(u-c)} - \frac{u''}{(u-c^*)}\right) f^* f dy \\ 0 &= \int_{-\infty}^{\infty} \left(\frac{u''}{(u-c)} - \frac{u''}{(u-c^*)}\right) f^* f dy \\ 0 &= \int_{-\infty}^{\infty} \left(\frac{u''(c-c^*)}{|u-c|^2}\right) |f|^2 dy. \end{aligned}$$

Note that for a complex function g, we have $g^*g = |g|^2$. We can now substitute $c = c_r + ic_i$ to obtain

$$\int_{-\infty}^{\infty} \left(\frac{2ic_i u''}{|u-c|^2}\right) |f|^2 dy = 0. \tag{11.57}$$

To have instability $c_i > 0$, thus u'' must change sign somewhere in the domain in order that (11.57) can equal zero. This observation that for an inviscid shear flow to be unstable, the velocity profile must have an inflection point (u'' changing sign) is known as Rayleigh's inflection-point theorem. It is a necessary, but not sufficient condition for inviscid shear flow instability.

11.2.3 Physical interpretation of Rayleigh's Theorem

The physical explanation for Rayleigh's inflection-point theorem is well presented by Lin (1966). For a two-dimensional shear flow, the vorticity distribution is given by

$$\xi(y) = -\frac{dU}{dy} \tag{11.58}$$

thus, when U has an inflection point, $d^2U/dy^2 = 0$ and the vorticity has a local maximum.

Consider now a fluid element as it approaches the vorticity maximum. If fluid with lower vorticity moves up to a region of higher vorticity the net feedback is to force the fluid back to its original location. Similarly, if fluid of high vorticity moves down into a region of lower vorticity, the net feedback forces the fluid back to the zone of higher vorticity. Thus, as long as the vorticity is monotonically increasing, the vorticity feedback provides stability. On the other hand, if a fluid packet of lower vorticity moves up to a local zone of maximum vorticity it is not forced back to its original location. In fact, it is equally at home on the other side of the vorticity maximum. Thus, mixing and exchange is easier and we may expect instability.

Summary

This chapter introduced the general concept of and analysis techniques for hydrodynamics instability. These methods were then applied to two examples. For the case of a stratified shear flow, a heuristic approach led to a critical Richardson number as the criteria for instability. The more detail stability analysis required solution of an Eigenvalue problem and resulted in a similar Richardson number criteria. The other case considered was a general shear flow without density stratification. Through derivation of the Rayleigh inflection-point theorem, we showed that a necessary (though not sufficient) condition for an inviscid shear flow to be unstable is that it have an inflection point.

Exercises

11.1 Rayleigh-Taylor Instability. Assume that a two-layer density-stratified system is unstably stratified ($\rho_1 > \rho_2$) and is initially stagnant ($U_0 = 0$). First use (11.35) to show that, under our assumptions, the system must be unstable.

Now include the effects of surface tension. Use (11.36) to show that the system is unstable for

$$(\rho_2 - \rho_1)g > \frac{\pi^2}{a^2}T. \tag{11.59}$$

You will need the dispersion relation for the normal modes of linear waves in a two layer system:

$$(\rho_1 + \rho_2)\omega_N^2 = \frac{N\pi}{a}\left[(\rho_1 - \rho_2)g + T\frac{N^2\pi^2}{a^2}\right] \tag{11.60}$$

where N is the node number and a is the wave amplitude.

Show that for water overlaying air, $a > 9$ mm is required for the onset of instability. This explains why water can be retained in an inverted glass if the mouth of the glass is covered by a fine-meshed gauze. [Adapted from Acheson (1990)].

11.2 Shear flow mixing layer. Determine the stability characteristics of an unstratified ($\rho_1 = \rho_2$) mixing layer ($|U_0| > 0$). Discuss this situation from both the K-H analysis and Rayleigh's inflection-point theorem perspective.

11.3 Groundwater stability. Using the governing equations for incompressible flow through a porous media:

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{11.61}$$

$$0 = -\frac{1}{\rho_f}\frac{\partial p}{\partial x_i} - \frac{\nu}{\kappa}u_i + g_i \tag{11.62}$$

where ρ_f is the fluid density and κ is the Darcy conductivity, deduce the stability criteria for a two-layer stratified shear flow.

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