Free Shear Flows [1]

Turbulent free shear flows are the simplest examples of inhomogeneous turbulent flows. Free shear flows can be defined as flows that evolve away from a boundary, and typically consists of a mean velocity profile evolving along the axial direction (e.g. a spreading jet). These flows typically display *self-similarity*, due to the absence of any imposed external length scale (e.g. from confinement effects). Self-similarity essentially means that the cross sectional mean velocity profile has the same shape, *regardless* of the axial location. At any axial location in a free shear flow, x, we can define 3 different quantities or "scales": Centerline velocity $U_c(x)$, shear velocity U_s, and width $\delta(x)$. Also, there is an ambient velocity U_∞. Some examples of free shear flows are (Fig. 1):

1. Wakes: In this case, $U_\infty > 0$, $U_s(x) = U_\infty - U_c(x)$, and we will see that $\delta(x) \propto x^{1/2}$ if $U_s \ll U_c$

2. Jets: For co-flowing jets, $U_\infty > 0$, while for jets in a quiescent environment, $U_\infty = 0$. Again, $U_s(x) = U_c(x) - U_\infty$. For jets in quiscent environment, $U_s = U_c$. For this case, we will see that $\delta(x) \propto x$

3. Mixing layer: Here, there is no unique U_∞; instead we have velocity going to U_1 and U_2 on both sides on the mixing layer. $U_s(x) = U_2 - U_1$ is a constant, and $U_c = \frac{U_1+U_2}{2}$, and $r = U_s/U_c$. We will require $U_s \ll U_c$ (or $r \ll 1$) to obtain a self-similar solution here, for which we can obtain $\delta(x) \propto x$.

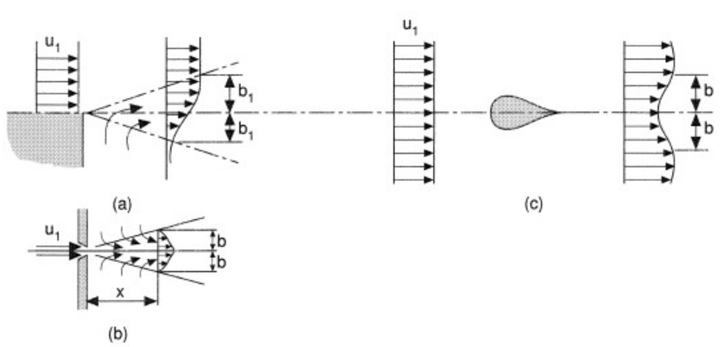

Figure 1: (a) Mixing layer (b) Jet (c) Wakes. Notation of quantities in figure is different from the text.

The "width" of a wake has to be defined consistently before beginning the analysis. In a jet, we can, for instance, define it as the y location where $U(x,y)|_{y=\delta(x)} = \frac{1}{2}U_c(x)$. There are many other ways to define the meaning of width $\delta(x)$, but, once we choose a definition, our analysis has to be consistent with it.

In our quest for a self-similar solution, we will start off with the equations for mean velocity, $U(x,y)$ and $V(x,y)$:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{1}$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right] - \frac{\partial \overline{u^2}}{\partial x} - \frac{\partial \overline{uv}}{\partial y} \tag{2}$$

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = -\frac{\partial P}{\partial y} + \nu\left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right] - \frac{\partial \overline{uv}}{\partial x} - \frac{\partial \overline{v^2}}{\partial y} \tag{3}$$

We now try and simplify the equations by performing "scaling analysis", in which we will first make assumptions about the relative size of scales, and then try to see which terms can, as a result, be neglected. The scales already mentioned are U_c, δ, U_s. One more important scale is $L \sim \frac{\partial U/\partial x}{\partial^2 U/\partial x^2}$, which gives an idea of the length scale over which the axial velocity is changing. Another scale is q, which is the size (e.g. standard deviation) of the turbulent fluctuations, u, v. To start with, it is easy to see that, at any x location:

$$U \sim U_c \tag{4}$$

$$\frac{\partial U}{\partial y} \sim \frac{U_s}{\delta} \tag{5}$$

However, the scale for V has to be obtained from the continuity equation:

$$\begin{array}{ccc} \frac{\partial U}{\partial x} + & \frac{\partial V}{\partial y} & = 0 \\ \left[\frac{U_s}{L}\right] & \left[\frac{V}{\delta}\right] & \end{array} \tag{6}$$

Implying that

$$V \sim U_s \frac{\delta}{L} \tag{7}$$

Next, to understand the scaling of pressure, we look at the V momentum equation (scalings multiplied by δ):

$$\begin{array}{cccccccc} U\frac{\partial V}{\partial x} + & V\frac{\partial V}{\partial y} + & \frac{\partial \overline{uv}}{\partial x} + & \frac{\partial \overline{v^2}}{\partial y} & = & -\frac{\partial P}{\partial y} & +\nu\left[\frac{\partial^2 V}{\partial x^2}\right. & \left.+\frac{\partial V}{\partial y^2}\right] \\ \left[U_c U_s \frac{\delta^2}{L^2}\right] & \left[U_s^2 \frac{\delta^2}{L^2}\right] & \left[q^2 \frac{\delta}{L}\right] & \left[q^2\right] & & [??] & \left[\nu \frac{U_s}{L}\frac{\delta^2}{L^2}\right] & \left[\nu \frac{U_s}{L}\right] \end{array} \tag{8}$$

For $Re = \rightarrow \infty$ and $\delta/L \rightarrow 0$, balance is possible only if:

$$-\frac{\partial P}{\partial y} = \frac{\partial \overline{v^2}}{\partial y} \tag{9}$$

The boundary conditions $\overline{vv}(x, \pm\infty) = P(x, \pm\infty) = 0$ implies (along with eqn (9)) that $\overline{v^2} = -P$ for the whole flow. Hence,

$$\frac{\partial P}{\partial x} = -\frac{\partial \overline{v^2}}{\partial x} \tag{10}$$

Substituting eqn (10) into eqn (2), we get the U momentum equation without P:

$$\begin{array}{cccccc} U\frac{\partial U}{\partial x}+ & V\frac{\partial U}{\partial y}+ & \frac{\partial \overline{u^2}-\overline{v^2}}{\partial x}+ & \frac{\partial \overline{uv}}{\partial y} & = \nu\left[\frac{\partial^2 U}{\partial x^2}\right. & \left.+\frac{\partial U}{\partial y^2}\right] \\ \left[\frac{U_c U_s}{L}\right] & \left[\frac{U_s\delta}{L}\frac{U_s}{\delta}\right] & \left[\frac{q^2}{L}\right] & \left[\frac{q^2}{\delta}\right] & \left[\frac{\nu U_s}{L^2}\right] & \left[\frac{\nu U_s}{\delta^2}\right] \end{array} \tag{11}$$

After considering the limit $\delta/L \rightarrow 0$, $\mathrm{Re}_\delta \rightarrow \infty$, we get (after multiplying the scaling by $\frac{\delta}{U_s^2}$):

$$\begin{array}{ccc} U\frac{\partial U}{\partial x}+ & V\frac{\partial U}{\partial y}+ & \frac{\partial \overline{uv}}{\partial y} = 0 \\ \left[\frac{U_c\delta}{U_s L}\right] & \left[\frac{\delta}{L}\right] & \left[\frac{q^2}{U_s^2}\right] \end{array} \tag{12}$$

There are now 2 cases we consider:

Case 1 $U_c \gg U_s$: This can occur in the far field of a wake or co-flowing jet (i.e. $|U_{cl} - U_\infty| \ll U_\infty$), or in a mixing layer with $r \ll 1$. In this case, clearly, the first term on the LHS will dominate over the 2nd term on the LHS, and will balance the Reynolds stress term (3rd term on LHS). Therefore, for this case, we can simplify the equation to:

$$U\frac{\partial U}{\partial x} + \frac{\partial \overline{uv}}{\partial y} = 0 \tag{13}$$

In these type of flows, we can typically express the velocity in the form $U(x, y) = U_0 + U_s(x, y)$, where $|U_0| \gg |U_s|$, and U_0 is a constant. Therefore, the above equation can be approximated to:

$$U_0\frac{\partial U_s}{\partial x} + \frac{\partial \overline{uv}}{\partial y} = 0 \tag{14}$$

The above equation gives the local momentum conservation equation at a (x, y) location. We can also write a global momentum conservation equation by integrating from $-\infty$ to ∞ over y, to obtain:

$$U_0 \frac{d}{dx} \int_{-\infty}^{\infty} [U - U_0] dy = 0 \tag{15}$$

or,

$$M = \int_{-\infty}^{\infty} U_0 [U - U_0] dy = \text{Constant} \tag{16}$$

where M is the net momentum flux per unit mass at any x location, and it has great physical significance. For instance, we can prove that, for a wake, if x is located in the far field, then $U_0 = U_\infty$ and:

$$\rho \int_{-\infty}^{\infty} U_\infty [U - U_\infty] dy = -D \tag{17}$$

where D is the drag of the bluff body around which the wake is being formed. The constraint given by eqn (17) will be useful for us when we are trying to derive a self-similar form of the velocity profile. Note that since U_0 is a constant, therefore it means that the net mass flux $\int_{-\infty}^{\infty} [U - U_0] dy$ is also a constant at any x location.

Case 2 $U_c \sim U_s$: This can occur for jets with no co-flow, near field of wakes and for mixing layers with $r \sim 1$. This case, the Reynolds stress term again has to balance the convective terms. Therefore, the only scaling possible is $q^2/U_s^2 \sim \delta/L$, and the equation cannot be simplified further, leading to the same momentum equation:

$$\int_{-\infty}^{\infty} \left[U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} \right] dy = 0 \tag{18}$$

Integrating the above equation in its conservative form, we obtain the global momentum equation:

$$\frac{d}{dx} \int_{-\infty}^{\infty} U^2 dy = -VU\Big|_{y=-\infty}^{y=\infty} \tag{19}$$

For a jet, $U(x, \pm\infty) = 0$, and therefore we can simply write:

$$\int_{-\infty}^{\infty} U^2 dy = \text{constant} = U_{in}^2 \Delta \tag{20}$$

where U_{in} is the velocity at the inlet and Δ is the width of the inlet. For a wake, we know that $U(x, \pm\infty) = U_\infty$. Therefore, using the integral form of the continuity equation:

$$\frac{d}{dx}\int_{-\infty}^{\infty} U dy = -V\Big|_{-\infty}^{\infty} \tag{21}$$

we can simplify eqn (19) to:

$$\int_{-\infty}^{\infty} U(U - U_\infty)dy = \text{constant} \tag{22}$$

The above equation is in fact a more exact version of eqn (16), since we are not assuming $U_s \ll U_c$ here. However, it is actually not possible to find a similarity form for the wake in the near field.

1 Similarity Analysis For the Wake

Here, we will try to obtain a similarity solution for a wake velocity profile in the far field, where we know that $U_s \ll U_c$ applies (case 1). Thus, we can approximate the wake velocity field as:

$$U(x, y) = U_\infty - \hat{u}(x, y) \tag{23}$$

Clearly, $\hat{u} \sim U_s$, and therefore $|\hat{u}| \ll U_\infty$. So, eqn (14) can be rewritten as:

$$U_\infty \frac{\partial \hat{u}}{\partial x} = \frac{\partial R_{12}}{\partial y} \tag{24}$$

Here, we have switched to the notation $\overline{uv} = R_{12}$ for convenience. The boundary conditions for $\hat{u}$ are:

$$\hat{u}(x, \pm\infty) = 0 \tag{25}$$

$$\frac{\partial \hat{u}}{\partial y}\Big|_{y=0} = 0 \tag{26}$$

The global momentum flux constraint (eqn (17)) can now be re-written as:

$$\int_0^{\infty} \rho U_\infty \hat{u} dy = \frac{1}{2}D \tag{27}$$

The mixing length model will be used to obtain closure for R_{12}:

$$R_{12} = -l_{mix}^2 \left|\frac{\partial U}{\partial y}\right| \frac{\partial U}{\partial y} = l_{mix}^2 \left|\frac{\partial \hat{u}}{\partial y}\right| \frac{\partial \hat{u}}{\partial y} \tag{28}$$

$$\text{where } l_{mix} = \alpha\delta \tag{29}$$

where α is a constant. This choice of l_{mix} is motivated by the fact that the largest eddies in the flow, at any x location, should be proportional to $\delta(x)$. The value of α can however vary between different types of flows, and is essentially decided by experiments.

Eqn (24) can be re-written, using eqn (28), as:

$$U_\infty \frac{\partial \hat{u}}{\partial x} = \alpha^2 \frac{\partial}{\partial y}\left[\delta^2 \left|\frac{\partial \hat{u}}{\partial y}\right| \frac{\partial \hat{u}}{\partial y}\right] \tag{30}$$

Now we are ready to find the self-similar solution for velocity. Self-similarity can actually be justified using dimensional analysis. To start with, we can write down a general expression for $\hat{u}(x, y)$:

$$\hat{u} = G(x, y, U_c(x), U_s(x), \delta(x), L(x), \nu) \tag{31}$$

Now, since there are only 2 dimensions (L, T) and 8 parameters, therefore, we should be able to reduce the above relation to 6 non-dimensional quantities:

$$\hat{u}(x, y) = U_s(x) G^*(x/\delta, y/\delta, U_c/U_s, \delta/L, \delta U_s/\nu) \tag{32}$$

In the limit of $U_c/U_s \to 0$, $\delta U_s/\nu \to \infty$, $\delta/L \to 0$ and $\delta/x \to 0$, we hope that G^* depends only on y/δ [2]. Thus, we can write the "self-similar" form:

$$\hat{u}(x, y) = u_0(x) F(\eta) \tag{33}$$

where $u_0(x)$ is some function of x, and $\eta = y/\delta(x)$. We would now like to know the form of $u_0(x)$ and $F(\eta)$. Towards that goal, we will substitute eqn (33) into eqns (24) and (27). Chain rule gives us the following rules for transforming between variables:

$$\left.\frac{\partial}{\partial x}\right|_y = \left.\frac{\partial}{\partial x}\right|_\eta + \left.\frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right|_x \tag{34}$$

$$\left.\frac{\partial}{\partial y}\right|_x = \left.\frac{\partial \eta}{\partial y}\right|_x \left.\frac{\partial}{\partial \eta}\right|_x \tag{35}$$

where we can derive:

$$\left.\frac{\partial \eta}{\partial x}\right|_y = -\frac{\delta'}{\delta}\eta \tag{36}$$

$$\left.\frac{\partial \eta}{\partial y}\right|_x = \frac{1}{\delta} \tag{37}$$

Thus:

$$\left.\frac{\partial \hat{u}}{\partial x}\right|_y = F(\eta) u_0'(x) - u_0(x) F'(\eta) \frac{\delta'(x)}{\delta(x)} \tag{38}$$

$$\left.\frac{\partial \hat{u}}{\partial y}\right|_x = \frac{u_0(x)}{\delta(x)} F'(\eta) \tag{39}$$

[2] This is actually a very strong assumption. For instance, if $G^* \propto (\delta/L)$, then $G^* \to 0$ for $\delta/L \to 0$.

The local and global momentum balance equations (eqns (30),(27)) can now be written as:

$$\left[\frac{\delta U_\infty u_0'}{u_0^2}\right] F(\eta) - \left[\frac{U_\infty \delta'}{u_0}\right] \eta F'(\eta) = \alpha^2 \frac{d}{d\eta}\left[|F'(\eta)|F'(\eta)\right] \quad (40)$$

$$[\rho U_\infty \delta u_0] \int_0^\infty F(\eta) d\eta = \frac{D}{2} \quad (41)$$

The above equations can be consistent only if the terms in the square brackets do not depend on x. Thus, we impose the constraints:

$$\frac{\delta U_\infty u_0'}{u_0^2} = a_1 \quad (42)$$

$$\frac{U_\infty \delta'}{u_0} = a_2 \quad (43)$$

$$\frac{D}{2\rho U_\infty u_0 \delta} = 1 \quad (44)$$

$$\int_0^\infty F(\eta) d\eta = 1 \quad (45)$$

The last constraint (eqn (45)) is kept to avoid an additional undetermined constant in eqn (44). Imposing this constraint does not have any other consequence, since any resulting factor is absobed into u_0. Now we can also easily show that the constraints in eqn (42)-43 are equivalent, since, from eqn (44), $u_0\delta$ is a constant and therefore $u_0\delta' = -\delta u_0'$. As a result, $a_1 = -a_2$. So in fact, we just need to satisfy eqns (43)–(44). Dividing eqn (43) by eqn (44), we obtain:

$$\frac{2U_\infty^2 \rho \delta' \delta}{D} = a_2 \quad (46)$$

$$\Rightarrow \delta(x) = \sqrt{\frac{D a_2 (x - x_0)}{U_\infty^2 \rho}} \quad (47)$$

Where x_0 is a constant denoting the "virtual origin" of the wake. Using eqn (47) and eqn (44), we can also say:

$$u_0(x) = \frac{1}{2}\sqrt{\frac{D}{a_2 \rho (x - x_0)}} \quad (48)$$

Effectively, we have been able to show that $u_0 \propto (x - x_0)^{-1/2}$ and $\delta \propto \sqrt{x - x_0}$. We substitute the above expressions for u_0 and δ into eqn (40), and obtain the following ODE:

$$a_2(F + F'\eta) - \alpha^2 \frac{d}{d\eta}[F']^2 = 0 \quad (49)$$

Here, we have used the simplification that, $F'(\eta) \leq 0$ for $\eta > 0$ (this is consistent with the final expression for $F(\eta)$ below). Moreover, based on the boundary conditions for $\hat{u}$ in eqns

(25)–(26), we can obtain:

$$F'(0) = 0 \quad F(\infty) \quad = \quad 0 \tag{50}$$

Using the equation for $F(\eta)$, along with the boundary conditions above, we can obtain the solution:

$$F(\eta) \quad = \quad \begin{cases} C^2 \left[1 - \left(\frac{\eta}{\eta_e}\right)^{3/2}\right]^2 & \text{for } \eta < \eta_e \\ 0 & \text{for } \eta > \eta_e \end{cases} \tag{51}$$

where $\eta_e = \left(\frac{3\alpha C}{\sqrt{a_2}}\right)^{2/3}$. We now have several constants (C, α, a_2, x_0), which need to be assigned in some way. First, we define δ as the y location where $\hat{u} = 0$, since it's easy to define this location for the above expression. This implies $F(1) = 0$, which in turn means that $\eta_e = 1$, or:

$$\left(\frac{3\alpha C}{\sqrt{a_2}}\right)^{2/3} \quad = \quad 1 \tag{52}$$

Secondly, we apply the constraint in eqn (45), which then gives us:

$$C^2 \int_0^1 [1 - \eta^{3/2}]^2 d\eta \quad = \quad 1 \tag{53}$$

The above constraints yield the following expressions for C and α:

$$C \quad = \quad 1.491 \tag{54}$$

$$\alpha \quad = \quad \sqrt{\frac{a_2}{20}} \tag{55}$$

The constraint in eqn (43) implies that the growth rate of the wake thickness, δ', when normalized w.r.t. u_0/U_∞ is a constant a_2. The value of a_2 is usually reported as the "spreading rate", and, typically, experiments imply $a_2 = 0.648$, and therefore, from eqn. (55), $\alpha = 0.18$. Finally, we get the following expression for the velocity field of the wake:

$$U(x, y) \quad = \quad U_\infty - 1.38\sqrt{\frac{D}{\rho(x - x_0)}} \left[1 - \eta^{3/2}\right]^2 \tag{56}$$

The value of x_0 is again usually obtained from experiments.

2 Obtaining scaling for $u_0(x)$ and $\delta(x)$ without solving for $F(\eta)$

It is clear from the above process that we did not really need to solve for $F(\eta)$ in order to obtain the scalings for $u_0(x)$, $\delta(x)$. In fact, we do not really even need to assume the mixing length model. Starting with the self similar form for u_0 in eqn (33), we can also assume a self-similar form for the Reynolds shear stress as $R_{12} = u_0^2(x)h(\eta)$, where $h(\eta)$ is some other function of η. Here, we are assuming that $q^2 \sim U_s^2$, which is reasonable, since we expect shear rate to give rise to turbulence. We substitute these self-similar forms in eqns (24) to obtain:

$$\left[\frac{\delta U_\infty u_0'}{u_0^2}\right] F(\eta) - \left[\frac{U_\infty \delta'}{u_0}\right] \eta F'(\eta) \quad = \quad h'(\eta) \tag{57}$$

Requiring the terms in the square brackets to be constants, and also applying the constraint in eqn (44) we can easily obtain the scaling $\delta \propto \sqrt{x - x_0}$ and $u_0 \propto \frac{1}{\sqrt{x}}$.

3 Entrainment of fluid

We can now ask, how much mass is entering the shear layer ? We can answer this by trying to estimate the mass flux $Q(x) = \int_{-\infty}^{\infty} U dy$ at any x location, and then seeing how dQ/dx varies with x. If $Q' > 0$, then clearly the shear layer is "entraining" fluid (i.e. mass is entering from the sides), and if $Q' < 0$, then the shear layer is "detraining" fluid (or mass is leaving the shear layer from the sides). For a self-similar profile, clearly, $Q(x) \propto u_0(x)\delta(x)$. Thus, for a wake, $Q(x)$ is a constant; it does not detrain or entrain fluid. Howver, for a jet, where we can show $u_0 \propto x^{-1/2}$ and $\delta \propto x$, we can see that $Q(x) \propto x^{1/2}$, and therefore a jet entrains fluid. Entrainment is an especially important phenomenon in atmospheric flows. For instance, a rising parcel of hot air (or "thermal bubble"), will stop rising at a low height if it manages to entrain a large amount of cold air very quickly.

4 Time and length scales of turbulence in shear flow

Clearly, the large scales in free shear flows have fluctuating velocity scale $q \sim U_s(x)$, length scale $\delta(x)$. Thus, the turbulent Reynolds number at any x location is $Re_\delta = \delta U_s/\nu$; clearly, for a wake, the Reynolds number does not change with x, while, for a jet, $Re_\delta \propto x^{1/2}$. We can also estimate the dissipation rate $\epsilon(x)$ as $\epsilon(x) \sim U_s^3/\delta$. Therefore, the Kolmogorov scale is $\eta = \left[\frac{\nu^3}{\epsilon}\right]^{1/4} = Re_\delta^{-3/4}\delta$. Thus, for both the wake and the jet, the Kolmogorov scale *increases* at larger x values. This implies that, when we perform numerical simulations, we can use a coarser resolution of the flow at larger x. However, we will also need a larger size of the box in the lateral (y, z) directions at larger values of x, and, in general, number of grid points in the lateral direction will be $\delta/\eta \propto Re_\delta^{3/4}$.