ME724:Turbulent Boundary Layers

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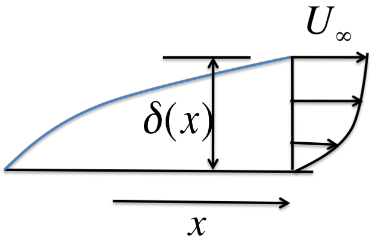
Turbulent Boundary Layers (TBL)

Text: "Boundary Layer Theory", Schlichting and Gersten

- y: Wall normal direction
- $\delta(x)$: Boundary layer thickness
- l : Characteristic length scale
- $U_\infty(x), P_\infty(x)$: Free stream velocity
- U, V change with both x, y
- Outside BL: $U_\infty \frac{\partial U_\infty}{\partial x} = -\frac{\partial P_\infty}{\partial x}$ (From Bernoulli's Equation)



Types of BL thickness

- General notation: $\delta(x)$
- Displacement thickness:

$$\delta^* = \frac{1}{U_\infty}\int_0^\delta (U_\infty - U)dy \quad (1)$$

- Momentum thickness:

$$\theta = \frac{1}{U_\infty^2}\int_0^\delta U(U_\infty - U)dy \quad (2)$$

Shape factor: $H = \delta^*/\theta$. $H \approx 1.3$ for turbulent flows, $H \approx 2.5$ for laminar flows.

δ^* vs δ

Rewrite expession for δ^* as:

$$\frac{\delta^* U_\infty}{u_\tau} = \frac{1}{u_\tau}\int_0^\delta (U_\infty - U)dy \tag{3}$$

In the outer region, deficit law implies:

$$U_\infty - U = u_\tau F(\eta) \tag{4}$$

where $\eta = y/\delta$. Thus:

$$\frac{\delta^* U_\infty}{u_\tau} \propto \delta \tag{5}$$

Exact TBL Equations

- Assumptions: $Re \to \infty$, $V \ll U$, $\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$, $R_{ij}(x, \delta) = 0$
- V momentum eqn:

$$\begin{aligned} 0 &= -\frac{\partial P}{\partial y} - \frac{\partial R_2 2}{\partial y} && (6) \\ \Rightarrow P(x,y) - P_\infty(x) &= -R_{22}(x,y) + R_{22}(x,\infty) && (7) \\ \Rightarrow \frac{\partial P}{\partial x} &= \frac{\partial P_\infty}{\partial x} - \frac{\partial R_{22}}{\partial x} && (8) \end{aligned}$$

- Since $\frac{\partial R_{22}}{\partial x} \ll \frac{\partial R_{12}}{\partial y} \ldots$

$$\begin{aligned} \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} &= 0 && (9) \\ U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} &= -\frac{\partial P_\infty}{\partial x} + \nu\frac{\partial^2 U}{\partial y^2} - \frac{\partial R_{12}}{\partial y} && (10) \end{aligned}$$

Momentum Integral Equation

After some manipulation, momentum equation can be written as:

$$\frac{\partial U(U - U_\infty)}{\partial x} + \frac{\partial V(U - U_\infty)}{\partial y} \tag{11}$$

$$+(U - U_\infty)\frac{\partial U_\infty}{\partial x} = \frac{\partial}{\partial y}\left[\nu\frac{\partial U}{\partial y} - R_{12}\right] \tag{12}$$

Integrating from $0 \rightarrow \infty$ in y, we get momentum integral equation:

$$\frac{1}{u_\tau^2}\frac{dU_\infty^2\theta}{dx} + \frac{U_\infty\delta^*}{u_\tau^2}\frac{dU_\infty}{dx} = 1 \tag{13}$$

Self-similarity

In general, for nonzero pressure gradient, we cannot expect a self-similar solution ($\frac{dP_\infty}{dx}$ introduces extra parameter). Self-similarity only possible if imposed $\frac{dP_\infty}{dx}$ is such that convective and Reynolds stress terms balance each other.

$$U\frac{\partial U_\infty - U}{\partial x} \sim U_\infty \frac{u_\tau}{L} \tag{14}$$

$$\frac{\partial R_{12}}{\partial y} \sim \frac{u_\tau^2}{\delta} \tag{15}$$

So, self-similarity possible only if $\frac{u_\tau^2}{\delta} \sim U_\infty \frac{u_\tau}{L}$, or if $\frac{U_\infty \delta}{L u_\tau}$ is a constant. Using previous relationships, we can alternately get the condition:

$$\frac{\delta^*}{\tau_w}\frac{dP}{dx} = \text{constant} \tag{16}$$

Equilibrium Boundary Layers

We can try and seek self-similar solution for the outer region (deficit law):

$$U^+(x,y) - U^+_\infty(x) = F(\eta) \tag{17}$$
$$R^+_{12} = G(\eta) \tag{18}$$

where $\eta = y/\delta$. Self-similarity possible, if $\frac{\delta^*}{\tau_w}\frac{dP}{dx} = \beta$=constant.

$$-2\beta F - (1+2\beta)\eta\frac{dF}{d\eta} = \frac{dG}{d\eta} \tag{19}$$

Boundary conditions:

$$F \to 0,\ G \to 0 \text{ for } \eta \to \infty,$$
$$G \to 1 \text{ for } \eta \to 0,$$
$$\eta F' = \frac{1}{\kappa} \text{ for } \eta \to 0$$

Equilibrium Boundary Layers

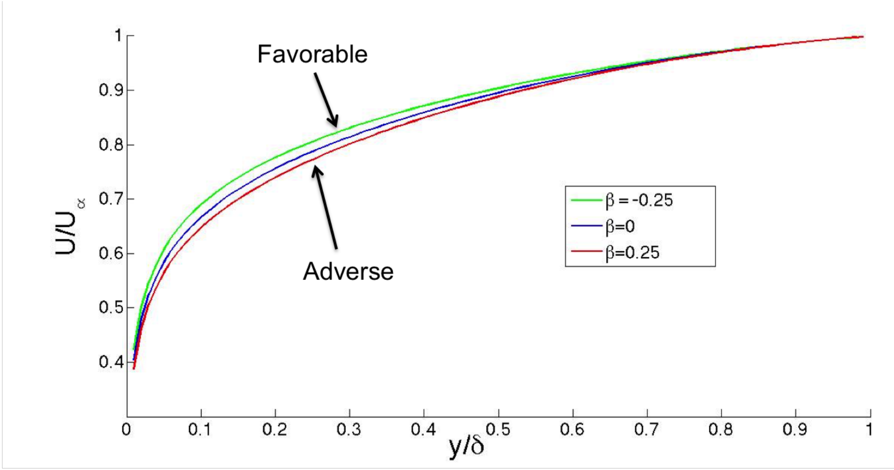
Clauser and Cole studied equilibrium TBL experimentally, and found the following relationship emipirically:

$$
\begin{aligned}
U^+_\infty(x) - U^+(y) &= -\frac{1}{\kappa}\log\eta + \frac{\Pi(\beta)}{\kappa}\left[2 - w(\eta)\right] \\
\text{where} \quad w(\eta) &= 2\sin^2\left[\frac{\pi}{2}\eta\right]
\end{aligned}
$$

$$
\Pi(\beta) = 0.8(\beta + 0.5)^{3/4} \qquad \beta = \frac{\delta^*}{\tau_w}\frac{dP}{dx}
$$

Adverse pressure gradient ($\frac{dP_\infty}{dx} > 0$) leads to higher deficit.

Equilibrium Boundary Layers



Equilibrium Boundary Layers

To find $C_f(x)$, start with deficit law in the limit $\eta \to 0$:

$$U_\infty^+(x) - U^+(x,y) = -\frac{1}{\kappa}\log\eta + \frac{2}{\kappa}\Pi(\beta) \tag{20}$$

$$\tag{21}$$

along with the "law of the wall" for the inner region:

$$U^+(x,y) = \frac{1}{\kappa}\log y^+ + B_i \tag{22}$$

Adding the two in the matching region ($\eta \to 0$, $y^+ \to \infty$):

$$U_\infty^+(x) = \frac{1}{\kappa}\log\left(\frac{Re_\delta(x)}{U_\infty^+}\right) + B_i + \frac{2}{\kappa}\Pi(\beta) \tag{23}$$

Using which we can solve for $C_f(x) = \frac{2}{(U_\infty^+)^2}$.

Zero Pressure Gradient TBL

For zero pressure gradient, $U_\infty(x)$ does not change with x. Momentum integral Eqn is:

$$\frac{1}{u_\tau^2}\frac{dU_\infty^2\theta}{dx} = 1 \tag{24}$$

$$\Rightarrow \frac{d\theta}{dx} = \frac{u_\tau^2}{U_\infty^2} = \frac{C_f}{2} \tag{25}$$

To solve this, we will try to make this into an ODE for $\delta(x)$. For skin friction:

$$C_f = 0.0205 Re_\delta^{-1/6} \tag{26}$$

To obtain momentum thickness $\theta(x)$ in terms of $\delta(x)$, use the well known approximation:

$$\frac{U(x,y)}{U_\infty(x)} = \eta^{1/7} \tag{27}$$

Does *not* give correct form of deficit law, but valid over large range of Re (we can instead use Coles and Clauser's wake parameter, but it will be cumbersome to integrate). Momentum thickness is:

$$\theta(x) = \frac{1}{U_\infty^2}\int_0^\delta U(U_\infty - U)dy = \delta\int_0^1 \eta^{1/7}(1-\eta^{1/7})d\eta = \frac{7}{72}\delta(x)$$

Zero Pressure Gradient TBL

Momentum integral equation becomes:

$$\frac{7}{72}\frac{d\delta}{dx} = 0.01025 Re_{\delta}^{-1/6} \tag{28}$$

$$\frac{7}{72}\delta^{1/6} d\delta = 0.01025 \left(\frac{\nu}{U_\infty}\right)^{1/6} dx \tag{29}$$

$$\Rightarrow \frac{\delta}{x} = 0.166 Re_x^{-1/7} \quad or \quad Re_\delta = 0.166 Re_x^{6/7} \tag{30}$$

Thus, BL grows like:

$$\delta \propto x^{6/7} \tag{31}$$

and

$$C_f(x) = 2\frac{d\theta}{dx} = 0.0277 Re_x^{-1/7} \tag{32}$$

Zero Pressure Gradient TBL

Coefficient of drag of a plate of length L is:

$$C_D(L) = \frac{1}{L}\int_0^L C_f(x)dx = 0.032Re_L^{-1/7} \qquad (33)$$

Examples

Relative importance of Viscous Drag Significant proportion of drag on an airplane comes from the fuselage. Say we approximate the fusealage as a cylinder with radius R and length L, with it's axis parallel to the flow. We can assume coefficient of form drag, C_p to be a constant (say $C_p \sim 0.1$).
Ratio of viscous to form drag force is then (assuming the fuselage surface is like a flat plate):

$$\frac{C_D(L)2\pi RL}{C_p\pi R^2} = \frac{2LC_D(L)}{C_pR} = \frac{2L}{R}\frac{0.032Re_L^{-1/7}}{0.1} \quad (34)$$

Typical airliner (Beoing/Airbus) will have $L/R \sim 10$, $L = 100$ meters, $R = 10m$, cruising speed of the plane is around 100m/s, and kinematic viscosity of air is $\nu = 10^{-5}m^2/s$, implying $Re_L \sim 10^9$. Thus, the fractional important of viscous drag is ~ 0.34; *most of the drag on an airplane is therefore from form drag*.

Examples

Relative importance of Viscous Drag Let's repeat the calculation for a truck: $U_\infty = 20$m/s, $L = 10$m, $R = 2m$, $\Rightarrow Re_L \sim 2 \times 10^7$,

$$\frac{C_D(L)2\pi RL}{C_p \pi R^2} = \frac{2LC_D(L)}{C_p R} = 10 \times \frac{0.032(2 \times 10^7)^{-1/7}}{0.1} = 2.89$$

Visocus drag is more important at lower Reynolds number (due to the $Re_L^{-1/7}$ dependence)

Examples

Thickness of TBL At the end of the fuselage, $\delta(L)$ will be given by:

$$\frac{\delta}{L} = 0.166 Re_L^{-1/7} \tag{35}$$

$$\Rightarrow \delta(L) = L \times 0.166 \times 10^{-9/7} = 0.83\mathrm{m} \tag{36}$$

Let's repeat the calculation for an automobile: $U_\infty = 20$m/s, $L = 4$m, $\Rightarrow Re_L \sim 10^7$,

$$\delta(L) = 4 \times 0.166 \times 10^{-1} = 0.07\mathrm{m} \tag{37}$$