Stability of Lorentz Attractor
ME724

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Lorenz Attractor: A Simple Chaotic System

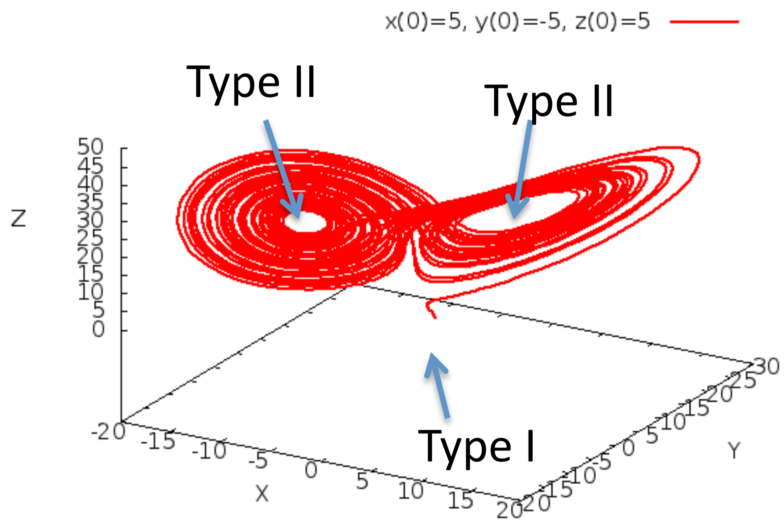
- Lumped-mass model for thermal convection

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = -xz + Rx - y$$
$$\dot{z} = xy - bz$$

  - R=Raleigh number, σ=Prandtl number, b=coupling constant

- Fixed points: $\dot{x} = \dot{y} = \dot{z} = 0$
  - Type I : $x = y = z = 0$
  - Type II : $x = y = \pm\sqrt{b(R-1)}$ $z = R - 1$
    - Exists only for $R>1$

Phase map of Lorenz attractor



Lorenz Attractor: Stability of Fixed Points

- Let x*=(x*,y*,z*) be fixed point
- Perturbation from fixed point: x(t)=x*+δx(t)
- Equation for perturbation

$$\dot{\mathbf{x}} = \mathbf{L}(\mathbf{x}) \quad \text{(Lorenz equation)} \Rightarrow \frac{d}{dt}[\mathbf{x}^* + \delta\mathbf{x}(t)] = \mathbf{L}(\mathbf{x}^* + \delta\mathbf{x}(t))$$

$$\frac{d}{dt}\delta\mathbf{x}(t) = \mathbf{L}(\mathbf{x}^*) + \nabla\mathbf{L}(\mathbf{x}^*)\delta\mathbf{x}(t) \quad \text{(Taylor Expansion)}$$

$$\Rightarrow \frac{d}{dt}\delta\mathbf{x}(t) = \nabla\mathbf{L}(\mathbf{x}^*)\delta\mathbf{x}(t) \quad \text{where } \nabla\mathbf{L} = \begin{bmatrix} L_{x,x} & L_{x,y} & L_{x,z} \\ L_{y,x} & L_{y,y} & L_{y,z} \\ L_{z,x} & L_{z,y} & L_{z,z} \end{bmatrix} = \overbrace{\begin{bmatrix} -\sigma & \sigma & 0 \\ R - z^* & -1 & x^* \\ y^* & x^* & -b \end{bmatrix}}^{\mathbf{A}}$$

- Eigenvalues of **A** determine stability of fixed point

Lorenz Attractor: Stability of Fixed Points

- If A has distinct eigenvalue and eigenvector pairs

$$\{(\mathbf{e}_i, \lambda_i); i = 1,2,3\} \text{ where } \mathbf{e}_i \in \mathrm{R}^3$$

$$\text{Then } \delta\mathbf{x}(t) = \sum_{i=1}^{3} \alpha_i \mathbf{e}_i \exp[\lambda_i t]$$

  - α_i determined from initial perturbation
  - If $\mathrm{Re}(\lambda_i)>0$ for any i, then fixed point is unstable

- Type I: $\lambda_1 = -b, \quad \lambda_{2,3} = -\frac{1}{2}\left[\frac{(1+\sigma) \pm \sqrt{(\sigma-1)^2 + 4R\sigma}}{2}\right]$

$$0 \le R < 1 \Rightarrow \lambda_{2,3} < 0$$

$$R = 1 \Rightarrow \lambda_2 = 0, \lambda_3 < 0$$

$$R > 1 \Rightarrow \lambda_2 > 0, \lambda_3 < 0$$

Fixed point unstable for $R > 1$

Stability of Fluid Flows

- N.S. equations for u(x,t):

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{N}(\mathbf{u})$$

- Choose laminar solution U(x,t) as fixed point
  - E.g. Parabolic profile for channel flow
- Equation for perturbation:

$$\frac{\partial \delta\mathbf{u}}{\partial t} = \nabla\mathbf{N}(\mathbf{U})\delta\mathbf{u}$$

- Find eigenvalues, eigenvectors of $\nabla\mathbf{N}(\mathbf{U})$
  - Above Re>Re_{cr}, we will find Re(λ)>0

Kelvin Helmholtz Instability