Dynamics of Two-Point Correlation and Energy Spectra

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Objectives

- Dynamics of the Spectrum Tensor
- Derivation of equation for two-point correlation $R_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x}) u_j(\mathbf{x}+\mathbf{r}) \rangle$ in homogeneous isotropic turbulence
- Relationship between the spectral energy transfer term and third order structure function
- Kolmogorov's 4/5ths law

Fourier Transforms (Greenberg, 17.10)

We have seen the Discrete Fourier Transform for a function defined over a periodic box $0 \leq x \leq L$:

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} \hat{f}_n \exp[ik_n x], \quad k_n = 2\pi n/L && (1)\\
\hat{f}_n &= \frac{1}{L}\int_0^L f(x)\exp[-ik_n x]\mathrm{d}x; \quad n \in \{0, \pm 1, \pm 2, \pm 3 \ldots\}
\end{aligned}
$$

In the limit of $L \to \infty$, $\Delta k = 2\pi/L \to 0$, and summation in eqn (1) can be converted to integral Fourier Transform (FT) and Inverse Fourier Transform (IFT):

$$
\begin{aligned}
IFT: \quad \mathcal{F}^{-1}[\hat{f}](x) = f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\exp[ikx]dx && (2)\\
FT: \mathcal{F}[f](k) = \hat{f}(k) &= \int_{-\infty}^{\infty} f(x)\exp[-ikx]dx && (3)
\end{aligned}
$$

Properties of Fourier Transforms (Greenberg, 17.10)

- Transform of Derivative: $\mathcal{F}[d^n f/d_x^n](k) = (ik)^n \hat{f}(k)$
- Transform of Integral: $\mathcal{F}[\int_{-\infty}^{x} f(\xi)d\xi](k) = \frac{1}{ik}\hat{f}(k)$
- Convolution theorem:

$$\begin{aligned}\text{Define Convolution} : (f \star g)(x) &= \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi \\ \mathcal{F}[(f \star g)](k) &= \hat{f}(k)\hat{g}(k)\end{aligned}$$

- Some common FT pairs:

$$\begin{aligned}\mathcal{F}[\delta(x-a)](k) &= \exp[-ika] \\ \mathcal{F}[H(x)](k) &= (i\omega)^{-1} \\ \mathcal{F}[\exp(-x^2)](k) &= \sqrt{\pi}\exp[-k^2/4] \\ \mathcal{F}[f(ax)](k) &= \frac{1}{a}\hat{f}(k/a)\end{aligned}$$

Fourier Transforms

FT methods are useful for solving PDEs, since they convert PDEs into ODEs

Example: Solving 1D diffusion eqn:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad (-\infty < x < \infty, 0 < t < \infty) \\ IC : u(x,0) &= f(x) \end{aligned}$$

After FT in x, we get the following ODE problem:

$$\begin{aligned} \frac{d\hat{u}}{dt} + \alpha^2 k^2 \hat{u}(k) &= 0 \\ \hat{u}(k,0) &= \hat{f}(k) \end{aligned}$$

Solution of this ODE is:

$$\hat{u}(k,t) = \hat{f}(k) \exp[-\alpha^2 k^2 t]$$

Fourier Transforms

Noting that $\mathcal{F}^{-1}[\exp(-\alpha^2 k^2 t)](x) = \frac{1}{2\alpha\sqrt{\pi t}} \exp[-x^2/(4\alpha^2 t)]$ and using convolution theorem:

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4\alpha^2 t}\right] d\xi$$

Velocity Spectrum Function/Tensor

Let's consider a statistically homogeneous 1D velocity field $u(x)$.
Two point correlaton is:

$$R_{uu}(r) \quad = \quad \langle u(x)u(x+r)\rangle$$

"Velocity Spectrum" function $\phi_{uu}(k)$ is simply the FT of $R_{uu}(r)$:

$$\begin{aligned} \phi_{uu}(k) &= \mathcal{F}[R_{uu}(x)](k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{uu}(r)\exp[-ikr]dr \\ \Rightarrow R_{uu}(r) &= \int_{-\infty}^{\infty} \phi_{uu}(k)\exp[ikr]dr \end{aligned}$$

Velocity Spectrum Function

Velocity spectrum *tensor* $\phi_{ij}(\mathbf{k})$ is:

$$\phi_{ij}(\mathbf{k}) = \frac{1}{2\pi^3} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} R_{ij}(\mathbf{r}) \exp[-i\mathbf{k} \cdot \mathbf{r}] dr_1 dr_2 dr_3$$

where $R_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x}) u_j(\mathbf{x} + \mathbf{r}) \rangle$ By definition:

$$\begin{aligned} \langle uu \rangle &= \int_{-\infty}^{\infty} \phi_{uu}(k) dk \\ R_{ij}(0) &= \int \phi_{ij}(\mathbf{k}) d^3\mathbf{k} \end{aligned}$$

Structure Function versus Velocity Spectrum

Say $\phi_{uu}(k) = C|k|^{-n}$, then

$$R_{uu}(k) = C\int_{-\infty}^{\infty} |k|^{-n} \exp[ikx]dk$$

which does not converge ($\lim_{k\to 0} |k|^{-n} = \infty$)

Let's consider the second order structure function instead:

$$\begin{aligned} S_2(r) &= \left\langle [u(x+r) - u(x)]^2 \right\rangle = 2\left\langle u^2 \right\rangle - 2\left\langle u(x)u(x+r) \right\rangle \\ &= 2\int_{-\infty}^{\infty} [1 - \exp(ikr)]\phi_{uu}(k)dk \\ &= C\int_{-\infty}^{\infty} [1 - \exp(ikr)]|k|^{-n}dk \\ &= 2Cr^{n-1}\int_{-\infty}^{\infty} [1 - \exp(i\alpha)]|\alpha|^{-n}d\alpha \end{aligned}$$

Which converges, if $1 < n < 3$. $n = 5/3$ implies $n - 1 = 2/3$:

Basis for connection between $5/3$rds law and $2/3$rds law

2-point Correlation Tensor (Isotropic Turbulence)

Under Isotropy, $R_{ij}(\mathbf{r})$ can be completely represented as:

$$R_{ij}(\mathbf{r}) = A(r)r_i r_j + B(r)\delta_{ij}$$

For longitudinal and transverse correlations:

$$f(r) = \frac{R_{\alpha\alpha}(r\mathbf{e}_\alpha)}{R_{\alpha\alpha}(0)} = \frac{A(r)r^2 + B(r)}{\langle u_\alpha^2 \rangle} \quad (4)$$

$$g(r) = \frac{R_{\beta\beta}(r\mathbf{e}_\alpha)}{R_{\beta\beta}(0)} = \frac{B(r)}{\langle u_\alpha^2 \rangle} \quad \beta \neq \alpha \quad (5)$$

Relating $A(r)$, $B(r)$ to $f(r)$, $g(r)$..

$$R_{ij}(\mathbf{r}) = \left[\frac{(f(r) - g(r))}{r^2} r_i r_j + g(r)\delta_{ij}\right] \langle u^2 \rangle$$

2-point Correlation Tensor (Isotropic Turbulence)

Next, note that:

$$\frac{\partial \langle u_i(\mathbf{x}) u_j(\mathbf{x}') \rangle}{\partial x'_j} = 0$$

Use $\mathbf{r} = \mathbf{x}' - \mathbf{x}$, and chain rule, to obtain:

$$\left.\frac{\partial}{\partial x'_j}\right|_{\mathbf{x}} = \frac{\partial}{\partial r_j} \Rightarrow \frac{\partial R_{ij}(\mathbf{r})}{\partial r_j} = 0$$

It can be similarly shown that $\frac{\partial R_{ij}(\mathbf{r})}{\partial r_i} = 0$. Applying this continuity constraint, we get:

$$g(r) = f(r) + \frac{1}{2} r f'(r)$$

So, $f(r)$ is enough to define $R_{ij}(\mathbf{r})$

2-point Correlation Tensor (Isotropic Turbulence)

Several length scales can be derived from $R_{ij}(\mathbf{r})$. Integral length scale:

$$\mathcal{L} = \int_0^\infty \frac{R_{11}(\mathbf{e}_1 r)}{R_{11}(0)} dr = \int_0^\infty f(r) dr$$

Taylor micro-scale (easier to measure experimentally):

$$\lambda = \left[-\frac{1}{f''(0)} \right]^{1/2}$$

Reynolds number based on Taylor number is often used: $Re_\lambda = K^{1/2}\lambda/\nu$. Possible to show that $Re_\lambda \propto Re_T^{1/2}$.

2-point Correlation Tensor (Isotropic Turbulence)

Taylor micro-scale:

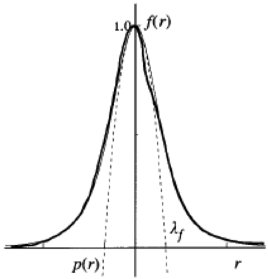

Fig. 6.7. A sketch of the longitudinal velocity autocorrelation function showing the definition of the Taylor microscale λ_f.

Equation for 2-point Correlation

- Assume no mean velocity field is present
- Let's start with the equation for fluctuating velocity $u_i(\mathbf{x})$:

$$\begin{aligned} \frac{\partial u_i}{\partial t} + \frac{\partial u_i u_k}{\partial x_k} &= -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} && (6) \\ \frac{\partial u_i}{\partial x_i} &= 0 && (7) \end{aligned}$$

- Also consider equation for velocity $u_i'(\mathbf{x}')$ at $\mathbf{x}' = \mathbf{x} + \mathbf{r}$:

$$\begin{aligned} \frac{\partial u_i'}{\partial t} + \frac{\partial u_i' u_k'}{\partial x_k'} &= -\frac{\partial p'}{\partial x_i'} + \nu \frac{\partial^2 u_i'}{\partial x_k' \partial x_k'} && (8) \\ \frac{\partial u_i'}{\partial x_i'} &= 0 && (9) \end{aligned}$$

Equation for 2-point Correlation

Let's first derive an equation for $\left\langle u_i(\mathbf{x})u_j'(\mathbf{x}')\right\rangle$:

- Taking average of $u_i \times \left[\frac{\partial u_j'}{\partial t} + ...\right] + u_j' \times \left[\frac{\partial u_i}{\partial t} + ...\right]$, we get:

$$\frac{\partial \left\langle u_i u_j' \right\rangle}{\partial t} + \frac{\partial \left\langle u_j' u_i u_k \right\rangle}{\partial x_k} + \frac{\partial \left\langle u_i u_j' u_k' \right\rangle}{\partial x_k'} = -\frac{\partial \left\langle u_j' p \right\rangle}{\partial x_i} - \frac{\partial \langle u_j p' \rangle}{\partial x_i'} + \nu \left[\frac{\partial^2 \left\langle u_i u_j' \right\rangle}{\partial x_k \partial x_k} + \frac{\partial^2 \left\langle u_i u_j' \right\rangle}{\partial x_k' \partial x_k'} \right]$$

Governing Equation for 2-Point Velocity Correlation
Homogenous Correlations

- In homogeneous turbulence, statistical quantities have the form $f(\mathbf{x}, \mathbf{x}') = F(\mathbf{r})$, where $\mathbf{r} = \mathbf{x}' - \mathbf{x}$
- From chain rule:

$$\frac{\partial}{\partial x_i} = \frac{\partial r_k}{\partial x_i}\frac{\partial}{\partial r_k} = -\delta_{ki}\frac{\partial}{\partial r_k} = -\frac{\partial}{\partial r_i} \quad (10)$$

$$\frac{\partial}{\partial x'_i} = \frac{\partial r_k}{\partial x'_i}\frac{\partial}{\partial r_k} = \delta_{ki}\frac{\partial}{\partial r_k} = \frac{\partial}{\partial r_i} \quad (11)$$

- Three correlations involved:
  - 2 point 3rd order velocity correlation: $R_{ij}(\mathbf{r}) = \left\langle u_i u'_j \right\rangle$
  - 2 point 3rd order velocity correlation: $S_{ikj}(\mathbf{r}) = \left\langle u'_j u_i u_k \right\rangle$
  - 2 point pressure-velocity correlation: $R_{pi}(\mathbf{r}) = \langle u'_i p \rangle$

Governing Equation for 2-Point Velocity Correlation Homogeneous Isotropic Turbulence

- Isotropy implies:

$$R_{ip}(\mathbf{r}) = 0 \quad (12)$$

$$S_{ikj}(\mathbf{r}) = A(r)[\delta_{ik} r_j + \delta_{jk} r_i] + B(r)\delta_{ij} r_k \quad (13)$$

- Equation for two-point correlation (under isotropy+homogeniety conditions):

$$\frac{\partial R_{ij}(\mathbf{r},t)}{\partial t} = \overbrace{\frac{\partial S_{ikj}}{\partial r_k}(\mathbf{r}) + \frac{\partial S_{jki}}{\partial r_k}(-\mathbf{r})}^{T_{ij}(\mathbf{r})} + 2\nu \frac{\partial^2 R_{ij}(\mathbf{r})}{\partial r_k \partial r_k}$$

$$\frac{\partial R_{ij}}{\partial r_i} = \frac{\partial R_{ij}}{\partial r_j} = 0 \quad (14)$$

Limit of zero separation

- Due to incompressibility, it can be shown that $\lim_{r\to 0} T_{ij}(\mathbf{r}) = 0$
- Also, $R_{ii}(0) = 2K = \langle u_i u_i \rangle$, implying

$$\frac{\partial K}{\partial t} = \nu \frac{\partial^2 R_{ii}}{\partial r_k \partial r_k}\bigg|_{\mathbf{r}=0} \qquad (15)$$

- Comparing with the kinetic energy equation ($\frac{dK}{dt} = -\epsilon$) derived earlier, we can say:

$$\nu \frac{\partial^2 R_{ii}}{\partial r_k \partial r_k}\bigg|_{\mathbf{r}=0} = -\epsilon \qquad (16)$$

- Using isotropy, and $\frac{d^2 R_{11}(r\mathbf{e}_1)}{dr^2} = -\frac{\langle u^2 \rangle}{\lambda^2}$ it can be shown that:

$$\epsilon = -15\nu \frac{\langle u^2 \rangle}{\lambda^2} \qquad (17)$$

Kolmogorov's 4/5th Law

The 2nd and 3rd order logitudinal structure function are given as:

$$S_L^2(r) = \left\langle (u_1(\mathbf{x}+r\mathbf{e}_1) - u_1(\mathbf{x}))^2 \right\rangle \quad (18)$$

$$S_L^3(r) = \left\langle (u_1(\mathbf{x}+r\mathbf{e}_1) - u_1(\mathbf{x}))^3 \right\rangle \quad (19)$$

From equation for 2-point correlation, it is possible to write:

$$\frac{3}{r^4}\int_0^r s^4 \frac{\partial S_L^2(s,t)}{\partial t} ds = 6\nu \frac{\partial S_L^2}{\partial r} - S_L^3 - \frac{4}{5}\epsilon r \quad (20)$$

In the inertial subrange, under steady state conditions,

$$S_L^3(r) = -\frac{4}{5}\epsilon r \quad (21)$$

This is one of the few exact analytical results in turbulence.

Isotropic Velocity Spectrum Tensor

Representation for $\phi_{ij}(\mathbf{k})$ in isotropic flows is:

$$\begin{aligned}
\phi_{ij}(\mathbf{k}) &= A(k)k_ik_j + B(k)\delta_{ij} \\
\frac{\partial R_{ij}}{\partial r_j} &= 0 \Rightarrow ik_j\phi_{ij} = 0 \\
\Rightarrow ik_j[A(k)k_ik_j + B(k)\delta_{ij}] &= 0 \\
\Rightarrow A(k)k^2 + B(k) &= 0 \Rightarrow \phi_{ij}(\mathbf{k}) = A(k)[k_ik_j - k^2\delta_{ij}]
\end{aligned}$$

Isotropic Velocity Spectrum Tensor

Now let's use:

$$\frac{1}{2}R_{kk}(0) = K = \int_0^\infty E(k)dk = \frac{1}{2}\int \phi_{kk}(\mathbf{k})d^3\mathbf{k}$$

But

$$\int \phi_{kk}(\mathbf{k})d^3\mathbf{k} = \int -2k^2 A(k)d^3\mathbf{k} = \int_0^\infty -2k^2 A(k)(4\pi k^2)dk$$

or, $A(k) = -\frac{1}{4\pi k^4}E(k)$, and therefore..

$$\phi_{ij}(\mathbf{k}) = \frac{E(k)}{4\pi k^4}\left[k^2\delta_{ij} - k_i k_j\right]$$

The "One Dimensional" Energy Spectrum

Typical experimental probes only measure just one velocity component in time. If the mean streamwise flow is U, and the turbulence is assumed to be "frozen" while passing through the probe then "Taylor's hypothesis" states:

$$\langle u(t)u(t+\Delta t)\rangle = R_{11}(\Delta t U \mathbf{e}_1)$$

How can we get the energy spectra $E(k)$ from the longitudinal 2-point correlation ? Start with the 1D energy spectrum $E_{11}(k_1)$:

$$E_{ij}(k_1) = \mathcal{F}[R_{ij}(r\mathbf{e}_1)](k_1)$$

But $R_{ij}(r\mathbf{e}_1) = \int \phi_{ij}(\mathbf{k}') \exp[ik_1' r] d^3\mathbf{k}'$, so that:

$$E_{ij}(k_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int \phi_{ij}(\mathbf{k}') \exp(ik_1' r) \exp(-ik_1 r) dk_1' dk_2' dk_3' \right] dr$$

The "One Dimensional" Energy Spectrum

Use $\delta(k_1 - k_1') = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \exp[-i(k_1 - k_1')]dr$ to get:

$$E_{ij}(k_1) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \phi_{ij}(\mathbf{k}')\delta(k_1 - k_1')dk_1'dk_2'dk_3'$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{E(k)}{4\pi k^4}(k^2\delta_{ij} - k_ik_j)dk_2dk_3$$

Specifically:

$$E_{11}(k_1) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{E(k)}{4\pi k^4}(k^2 - k_1^2)dk_2dk_3$$

The "One Dimensional" Energy Spectrum

Transforming variables to $k_2 = k_r \cos\theta$ and $k_3 = k_r \sin\theta$, we get:

$$E_{11}(k_1) = \int_{k_1}^{\infty} \frac{E(k)}{2k} \left[1 - \frac{k_1^2}{k^2}\right] dk$$

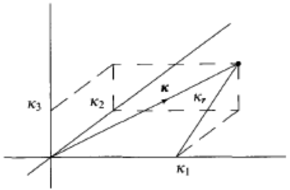

Fig. 6.10. A sketch of wavenumber space showing the definition of the radial coordinate κ_r.

The "One Dimensional" Energy Spectrum

Taking derivatives:

$$E(k_1) \;=\; k_1^3 \frac{d}{dk_1}\left(\frac{1}{k_1}\frac{dE_{11}}{dk_1}\right)$$

Clearly, $E(k) \propto \epsilon^{2/3} k^{-5/3}$ implies $E(k_1) \propto \epsilon^{2/3} k_1^{-5/3}$. Also,

$$\phi_{ij} \propto \frac{E(k)}{k^2} \sim \epsilon^{2/3} k^{-11/3}$$

The "One Dimensional" Energy Spectrum

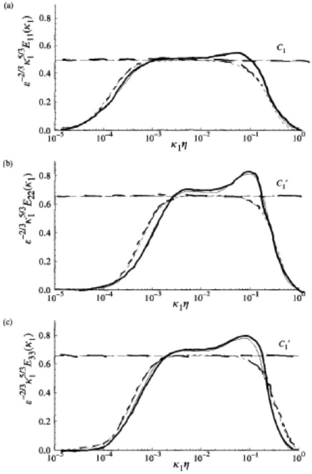

Fig. 6.17. Compensated one-dimensional spectra measured in a turbulent boundary layer at $R_\lambda \approx 1{,}450$. Solid lines, experimental data Saddoughi and Veeravalli (1994); dashed lines, model spectra from Eq. (6.246); long dashed lines, C_1 and C_1' corresponding to Kolmogorov inertial-range spectra. (For E_{11}, E_{22} and E_{33} the model spectra are for R_λ = 1,450, 690, and 910, respectively, corresponding to the measured values of $\langle u_1^2\rangle$, $\langle u_2^2\rangle$, and $\langle u_3^2\rangle$.)

Energy Transfer Via Triad Interactions

In a periodic box, 3D velocity field given by:

$$\mathbf{u}(\mathbf{x},t) = \sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k},t)\exp[i\mathbf{k}\cdot\mathbf{x}]$$

In wave-space, the NS equations are:

$$\begin{aligned}\left(\frac{d}{dt}+\nu k^2\right)\hat{u}_i &= -ik_l P_{jk}(\mathbf{k}) \sum_{\mathbf{k}',\mathbf{k}''} \underbrace{\delta_{\mathbf{k},\mathbf{k}'+\mathbf{k}''}\hat{u}_k(\mathbf{k}',t)\hat{u}_l(\mathbf{k}'',t)}_{\text{Triad Interactions}} + \\ &\quad \underbrace{\hat{f}_i(\mathbf{k},t)}_{\text{Large Scale Forcing}} \\ P_{jk}(\mathbf{k}) &= \delta_{jk} - \frac{k_j k_k}{k^2} \\ ik_k\hat{f}_k &= 0 \quad \text{(to ensure continuity)}\end{aligned}$$

Energy Transfer Via Triad Interactions

We can write down the equation for $\hat{E}(\mathbf{k},t) = \frac{1}{2}\left\langle \hat{u}_k \hat{u}_k^*(\mathbf{k},t)\right\rangle$ as:

$$\frac{d\hat{E}}{dt} = \mathrm{Real}[\hat{f}_k u_k^*(\mathbf{k},t)] + \hat{T}(\mathbf{k},t) - 2\nu k^2 \hat{E}(\mathbf{k},t) \quad (22)$$

$$\hat{T}(\mathbf{k},t) = k_l P_{jk} \mathrm{Real}\left[i \sum_{\mathbf{k}'} \left\langle \hat{u}_j(\mathbf{k}) \hat{u}_k^*(\mathbf{k}') \hat{u}_l^*(\mathbf{k}-\mathbf{k}')\right\rangle \right] \quad (23)$$

Possible to show that $\sum_{\mathbf{k}} \hat{T}(\mathbf{k},t) = 0$. Thus, $\hat{T}(\mathbf{k},t)$ does not add/subtract any energy globally. It only causes *transfer* of energy between scales.

Let's perform similar analysis via Fourier Transform

Evolution of Velocity Spectrum Tensor

- Obtain equation for velocity spectrum tensor by Fourier transforming two-point correlation tensor equation
- Remember that:

$$\phi_{ij}(\mathbf{k}) = \frac{1}{2\pi^3} \int R_{ij}(\mathbf{r}) \exp[-i\mathbf{k}\cdot\mathbf{r}] d\mathbf{r} \quad (24)$$

$$\Gamma_{ij}(\mathbf{k}) = \frac{1}{2\pi^3} \int T_{ij}(\mathbf{r}) \exp[-i\mathbf{k}\cdot\mathbf{r}] d\mathbf{r} \quad (25)$$

$$-k^2\phi_{ij}(\mathbf{k}) = \frac{1}{2\pi^3} \int \frac{\partial^2 R_{ij}(\mathbf{r})}{\partial r_k \partial r_k} \exp[-i\mathbf{k}\cdot\mathbf{r}] d\mathbf{r} \quad (26)$$

where $k^2 = k_i k_i$

- Let's also account for an average production, $P_{ij}(\mathbf{k})$, coming from a forcing term. This term is zero for k lying within inertial range and dissipative range.

Evolution of Velocity Spectrum Tensor

- Therefore:

$$\frac{\partial \phi_{ij}}{\partial t} = P_{ij} + \Gamma_{ij} - 2\nu k^2 \phi_{ij} \tag{27}$$

- Recalling that, for isotropic turbulence, $\phi_{ii}(\mathbf{k}) = E(k)/(2\pi k^2)$, we get

$$\frac{\partial E(k)}{\partial t} = P(k) + T(k) - 2\nu k^2 E(k) \tag{28}$$

  where
  - $P(k) = 2\pi k^2 P_{ii}(\mathbf{k})$ is the production. $P(k) > 0$ can be assumed.
  - $T(k) = 2\pi k^2 \Gamma_{ii}(\mathbf{k})$ can be seen as the energy input into the local k scale from the other scales.

Evolution of Energy Spectrum Tensor

- Integrating over k from 0 to ∞, and observing that $TKE = K = \int_0^\infty E(k)dk$, $\text{Net Production} = \mathcal{P} = \int_0^\infty P(k)dk$, $\int \Gamma_{ii}(\mathbf{k})d\mathbf{k} = T_{ii}(0) = 0$, we get:

$$\frac{dK}{dt} = \mathcal{P} - 2\nu \int_0^\infty k^2 E(k)dk \tag{29}$$

$$\Rightarrow \epsilon = 2\nu \int_0^\infty k^2 E(k)dk = \int_0^\infty D(k)dk \tag{30}$$

where $D(k) = 2\nu k^2 E(k)$ is the dissipation spectra

Some observations

- The net contribution of $T(k)$ to kinetic energy is zero: $\int_0^\infty T(k)dk$ (i.e. it just transports energy from one scale to another).
- If $E(k) \sim k^{-n}$ and $0 < n < 2$, then, for $k_1 \ll k_2$, we have

$$E(k_1) \gg E(k_2) \tag{31}$$

$$D(k_1) \ll D(k_2) \tag{32}$$

Consistent with our assertion that dissipation occurs in the small scales.

Some observations

- For statistically stationary turbulence, $\frac{\partial E(k)}{\partial t} = 0 = P(k) + T(k) - D(k)$
- $P(k), D(k)$ is small in inertial range
  - And, therefore, so is $T(k)$
  - In inertial scales, whatever energy that gets received from the large scales is passed on to the smaller scales
- $T(k) = -P(k) < 0$ for very small k (integral scales) and $T(k) = D(k) > 0$ for very large k (dissipative scales)
  - Energy needs to cascade from large to small scales

Energy transfer spectrum

- Now let's integrate the energy spectrum equation:

$$\frac{\partial \int_0^k E(k')dk'}{\partial t} = \int_0^k P(k')dk' + \int_0^k T(k')dk' - 2\nu \int_0^k k'^2 E(k')dk' \qquad (33)$$

- Clearly, $\Pi(k) = -\int_0^k T(k')dk'$ is the net transfer of energy from all scales with wavenumber less than k to all scales with wavenumber more than k.
- $\Pi(k)$ is called the "transfer" spectrum.
- For k in inertial range,
  - $\int_0^k P(k')dk' = \mathcal{P} = \epsilon$
  - and therefore $\Pi(k) = \epsilon$

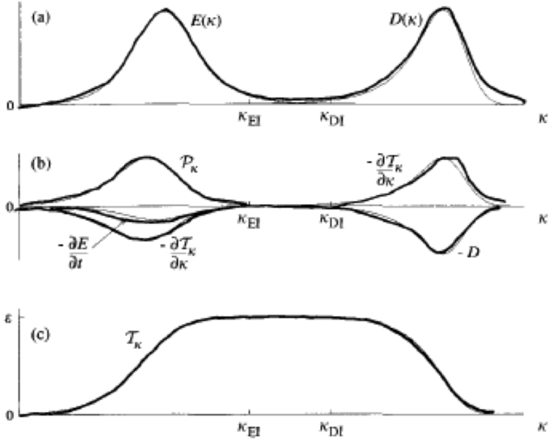

Fig. 6.28. For homogeneous turbulence at very high Reynolds number, sketches of (a) the energy and dissipation spectra, (b) the contributions to the balance equation for $E(\kappa, t)$ (Eq. (6.284)), and (c) the spectral energy-transfer rate.