ME741 – Scales of Turbulence, K41 Theory [1]

As evident from visualizations, turbulent flows are seen to contain a wide range of eddies, which are swirling around at different angular velocities. Typically, the larger eddies are swirling slowly, while the smaller ones seem to be swirling quickly. To try and develop a concrete theory about the structure of turbulence, we have to formally quantify "eddies" and "swirling". This is not a trivial task, and for certain types of flows (e.g. wall-bounded flows), this is still a topic of ongoing research. However, for most statistically stationary free shear flows (e.g. jets, mixing layers) at high Re, we can assume a condition called "local isotropy". By which, it is meant that, if we focus on a region much smaller than the main length scale of the flow (e.g. width of jet or mixing layer), then we can assume that the small eddies within that region behave like they would in isotropic turbulence. This hypothesis forms the basis of many LES and even RANS models. Therefore, we try and study isotropic turbulence in some detail in this chapter.

To be specific, we consider the case of forced 3D homogeneous (statistically) isotropic turbulence with periodic boundary conditions at the box limits. Physically, "forcing" simply means stirring the large scales in the flow externally (e.g. with a spoon). Periodicity allows us to decompose the velocity field into its respective Fourier modes. We first illustrate the relationship between Energy spectra and two-point correlations.

For pedagogical reasons, we first consider a scalar function $u(x)$ defined over 1D, $0 < x \leq L$, with periodic boundary conditions (so that $\partial_x^n u(0) = \partial_x^n u(L)$). For such a function, we can use the Fourier expansion:

$$u(x) \quad = \quad \sum_{j=-N/2}^{N/2-1} \hat{\hat{u}}_j \exp[\frac{i2\pi jx}{L}]. \tag{1}$$

Here, $\hat{\hat{u}}_j$ is the j^{th} Fourier component of the function (which can be a complex number), N is a positive integer, denoting total number of modes. The j^{th} Fourier mode then clearly represents the amplitude and phase of a 1D wave with wavelength $\lambda_j = L/j$; that is, we have decomposed the function into waves, of different wavelengths, each having a different phase. The above decomposition is more commonly written as $u(x) = \sum_{j=-N/2}^{N/2} \hat{u}(\kappa(j)) \exp[\frac{i\kappa(j)x}{L}]$, where $\hat{u}(\kappa(j)) = \hat{\hat{u}}_j$, and $\kappa(j) = 2\pi j/L = 2\pi/\lambda_j$ is known as the "wavenumber" of the Fourier mode. Eqn. (1) is now simply written as:

$$u(x) \quad = \quad \sum_{\kappa} \hat{u}(\kappa) \exp[i\kappa x], \tag{2}$$

in which it is understood that the summation is over all $\kappa(j) = 2\pi j/L$; $j = -N/2, ..., N/2 - 1$. Note that, for a given mode j, wavenumber κ_j is proportional to the inverse of wavelength λ_j. Thus, when we say we are looking at some property (say, amplitude) of small wavenumber modes, then it imples that we are looking at the property of large flow structures (i.e. modes

with large wavelength), and vice versa. In this theory, we assume that the range of wavenumbers is large (i.e. that N is large).

Our next observation is that, since $u(x)$ is real, therefore, $\hat{u}(k) = \hat{u}^*(-k)$. Furthermore, the Fourier modes are orthogonal, i.e.:

$$\int_0^L \exp[i\kappa x] \exp[i\kappa' x] \mathrm{d}x \quad = \quad L\delta_{\kappa,-\kappa'}, \tag{3}$$

where δ is a version of the Kronecker Delta function:

$$\delta_{\alpha,\beta} \quad = \quad \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{Otherwise} \end{cases} \tag{4}$$

Multiplying Eqn. (2) by $\exp[i\kappa' x]$, integrating over x between $[0\ L]$ and then applying the orthogonailty condition (3) yields:

$$\hat{u}(\kappa) = \frac{1}{L} \int_0^L u(x) \exp[i\kappa x] \mathrm{d}x, \tag{5}$$

which simply relates the Fourier component to the integral of the physical function.

Now we can show the relationship between correlations in physical space and the energy distribution amongst different wave-numbers (or energy spectra). Here we keep in mind that, since x is a direction of statistical homogeniety, therefore the average for any function $f(u)$ can be defined simply as $\overline{f(u)} = \frac{1}{L}\int_0^L \overline{f(u)} \mathrm{d}x$. Note that the averaging operator has been applied to the integrand as well, since we are applying ensemble averaging on top of spatial averaging. Thus, we obtain:

$$\overline{u} \quad = \quad \frac{1}{L} \int_0^L \overline{u(x)} \mathrm{d}x = \sum_\kappa \frac{1}{L} \overline{\hat{u}}(\kappa) \int_0^L \exp[i\kappa x] \mathrm{d}x = \overline{\hat{u}}(0). \tag{6}$$

The last equality follows from the fact that the integral of $\exp[i\kappa x]$ over x will be zero for $\kappa \neq 0$ [2]. The above equation states that the zeroth Fourier mode represents the mean velocity of the flow. While studying forced homogeneous isotropic turbulence, the mean plays no role in the dynamics and is equated to zero. Similarly, we can write down, for the energy:

$$\begin{aligned} \overline{u^2} \quad &= \quad \frac{1}{L} \int_0^L \overline{u(x)^2} \mathrm{d}x = \sum_\kappa \sum_{\kappa'} \overline{\hat{u}(\kappa)\hat{u}(\kappa')} \left[\frac{1}{L} \int_0^L \exp\{i\kappa x\} \exp\{i\kappa' x\} \mathrm{d}x \right] \\ &= \quad \sum_\kappa \sum_{\kappa'} \overline{\hat{u}(\kappa)\hat{u}(\kappa')} \delta_{\kappa,-\kappa'} \quad \text{(From orthogonality)} \\ \Rightarrow \overline{u^2} \quad &= \quad \sum_\kappa \overline{\hat{u}(\kappa)\hat{u}^*(\kappa)} = \sum_\kappa \overline{|\hat{u}(\kappa)|^2} \end{aligned} \tag{7}$$

where we have used $\hat{u}(-\kappa) = \hat{u}^*(\kappa)$. The above relationship implies that the energy of the flow can be partitioned into the energy of its individual Fourier components; this partitioning is possible mainly because of the orthogonality of the Fourier modes. In 1D, the quantity $E(\kappa) = \overline{|\hat{u}(\kappa)|^2}$ can be called as the energy spectra. We now turn to two-point correlations,

[2] keeping in mind that κ is only allowed to take the values $2\pi j/L$, where j is an integer

and derive:

$$\begin{aligned} R(r) &= \overline{u(x)u(x+r)} = \frac{1}{L}\int_0^L \overline{u(x)u(x+r)}\mathrm{d}x \\ &= \sum_\kappa \sum_{\kappa'} \overline{\hat{u}(\kappa)\hat{u}(\kappa')} \left[\frac{1}{L}\int_0^L \exp\{i\kappa x\}\exp\{i\kappa'(x+r)\}\mathrm{d}x\right] \\ &= \sum_\kappa \sum_{\kappa'} \overline{\hat{u}(\kappa)\hat{u}(\kappa')}\exp\{i\kappa' r\}\left[\frac{1}{L}\int_0^L \exp\{i\kappa x\}\exp\{i\kappa'(x)\}\mathrm{d}x\right] \\ \Rightarrow R(r) &= \sum_\kappa \overline{|\hat{u}(\kappa)|^2}\exp[i\kappa r] = \sum_\kappa E(\kappa)\exp[i\kappa r] \end{aligned} \tag{8}$$

The above equation implies that the energy spectra $E(\kappa)$ and two-point correlation $R(r)$ are Fourier transform pairs of each other. Using the fact that $R(r)$ is a real function, we can rewrite Eqn (8) as:

$$R(r) = \sum_{\kappa>0} E(\kappa)\{\exp[i\kappa r] + \exp[-i\kappa r]\} = \sum_{\kappa>0} 2E(\kappa)\cos(\kappa r) \tag{9}$$

which leads us to the following relation:

$$\Delta R(r) = R(0) - R(r) = \sum_{\kappa>0} 2E(\kappa)[1-\cos(\kappa r)] \tag{10}$$

In the above equation, Fourier modes with $\kappa \ll 1/r$ (and therefore with wavelengths much larger than r) cannot contribute to $\Delta R(r)$, since the factor $[1-\cos(\kappa r)] \approx 0$ for those modes. In other words, $\Delta R(r)$ is a measure of energy in scales equal to or smaller than r. We can also arrive at this result a bit more intuitively. Let's observe that :

$$\begin{aligned} \overline{(u(x')-u(x))^2} &= \left[\overline{u(x)u(x)} + \overline{u(x')u(x')} - 2\overline{u(x)u(x')}\right] = 2\left[\overline{u(x)u(x)} - \overline{u(x)u(x+r)}\right] \\ &= 2\left[R(0) - R(r)\right] = 2\Delta R(r) \end{aligned} \tag{11}$$

Thus, $\Delta R(r)$ represents a measure of velocity difference $\delta u_r = u(x+r) - u(x)$ between points that are separated by a distance r. Modes with wavelengths much larger than r will (by definition) hardly vary between x and $x+r$. Hence, δu_r (and therefore $\Delta R(r)$), will not get any contribution from these large-wavelength modes.

All these results can now be seen in 3D as well. In that case, the domain is a cubic box of size $L \times L \times L$, and the Fourier decomposition looks like:

$$u_\alpha(\mathbf{x}) = \sum_{\boldsymbol{\kappa}} \hat{u}_\alpha(\boldsymbol{\kappa})\exp[i\boldsymbol{\kappa}\cdot\mathbf{x}] \tag{12}$$

where $\boldsymbol{\kappa} = \kappa_1\hat{\mathbf{e}}_1 + \kappa_2\hat{\mathbf{e}}_2 + \kappa_3\hat{\mathbf{e}}_3$ is now the wave-vector, $\hat{\mathbf{e}}_i$ being the unit vector oriented along the i^{th} axis. The components κ_i of the wave-vector can take the values $\kappa_i(j) = 2\pi j/L$, with $j = -N/2, ..., 0, ...N/2-1$. The summation above is therefore over all the possible N^3 values that $\boldsymbol{\kappa}$ can take. To understand the physical meaning of the 3D Fourier mode $\exp[i\boldsymbol{\kappa}\cdot\mathbf{x}]$, consider the magnitude of the wave-vector, $\kappa = |\boldsymbol{\kappa}|$ and its orientation $\hat{\mathbf{e}} = \boldsymbol{\kappa}/\kappa$. We can express $\mathbf{x} = s\hat{\mathbf{e}} + n\hat{\mathbf{e}}^n$, where $\hat{\mathbf{e}}^n$ is some vector normal to $\hat{\mathbf{e}}$. Clearly, $\exp[i\boldsymbol{\kappa}\cdot\mathbf{x}] = \exp[i\kappa s]$ will be a constant along any $\hat{\mathbf{e}}^n$, while it will vary along $\hat{\mathbf{e}}$ like a one-dimensional Fourier wave.

In other words, $\exp[i\boldsymbol{\kappa}\cdot\mathbf{x}]$ is a Fourier wave traveling along $\boldsymbol{\kappa}$, with the "crests" of the wave oriented perpendicular to $\boldsymbol{\kappa}$.

In 3D, the average energy is given by:

$$K = \frac{1}{2}\overline{u_i(\mathbf{x})u_i(\mathbf{x})} \quad = \quad \frac{1}{2}\sum_{\boldsymbol{\kappa}} \overline{\hat{u}_i(\boldsymbol{\kappa})\hat{u}_i^*(\boldsymbol{\kappa})} \tag{13}$$

In the above equation, we have used the "Cartesian sum" notation, in which repeated indices indicates summation from 1 to 3 over those components (e.g. $a_i a_i = a_1a_1 + a_2a_2 + a_3a_3$). The quantity $\overline{\hat{u}_i(\boldsymbol{\kappa})\hat{u}_i^*(\boldsymbol{\kappa})}$ is called the correlation spectra (and *not* energy spectra); notice that it is a function of $\boldsymbol{\kappa}$. In isotropic turbulence, the energy per wavenumber should not depend on the direction of the wavenumber, therefore we would now like to define a quantity called energy spectra, $E(\kappa)$, such that $K = \int_0^\infty E(\kappa)\mathrm{d}\kappa$. We cannot convert the summation in Eqn (13) to a Reimann sum. Instead, we define:

$$E(\kappa) = \frac{1}{2}\sum_{\boldsymbol{\kappa}'} \overline{\hat{u}_i(\boldsymbol{\kappa}')\hat{u}_i^*(\boldsymbol{\kappa}')}\delta(\kappa - |\boldsymbol{\kappa}'|) \tag{14}$$

in which $\delta(x)$ is simply the Dirac delta function. Physically, Eqn (14) defines the energy spectra at wavenumber κ as the sum of the energy of wavenumbers whose magnitude is equal to κ. More importantly, $\int_{\kappa_a}^{\kappa_b} E(\kappa)\mathrm{d}\kappa$ gives the amount of energy present between wavenumbers $\kappa_a < \kappa < \kappa_b$. In 3D space, the 2^{nd} order, longitudinal "Structure function", $S_2(r) = \overline{(u_1(\mathbf{x} + r\hat{\mathbf{e}}_1) - u_1(\mathbf{x}))^2}$ is used to obtain scale-specific information about the flow field (similar to $\Delta R(r)$ earlier). It can be shown that (Pope, page 226):

$$S_2(r) \quad = \quad 2\int_0^\infty E_{11}(\kappa_1)[1 - \cos(\kappa_1 r)]\mathrm{d}\kappa_1 \tag{15}$$

$$\text{where} \quad E_{11}(\kappa_1) \quad = \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{E(\kappa)}{2\pi\kappa^2}\left[1 - \frac{\kappa_1^2}{\kappa^2}\right]\mathrm{d}\kappa_2\mathrm{d}\kappa_3 \tag{16}$$

where $\kappa = \sqrt{\kappa_1^2 + \kappa_2^2 + \kappa_3^2}$ and the integration is over a plane of fixed κ_1. $E_{11}(\kappa)$ is called the "one-dimensional" spectra, and is different from the energy spectra $E(\kappa)$. The important conclusion from Eqns (15) is that if $E(\kappa) \sim \kappa^{-n}$ then it follows that $S_2(r) \sim r^{n-1}$, and vice-versa. That is, the Structure function and Energy spectra are closely related to each other. Now, let us try to derive the value of exponent n. We start with the forced N.S. equation:

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_k}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_k \partial x_k} + f_i \tag{17}$$

where $\mathbf{f}(\mathbf{x}, t)$ is the forcing due to the "spoon", which is active only in the large scales (or low wavenmbers). We then derive the equation for the kinetic energy, $K = \overline{u_i u_i}/2$, by contracting Eqn (17) with u_i and then averaging:

$$\overline{u_i\frac{\partial u_i}{\partial t}} + \overline{u_i\frac{\partial u_i u_k}{\partial x_k}} = -\overline{u_i\frac{\partial p}{\partial x_i}} + \overline{\nu u_i\frac{\partial^2 u_i}{\partial x_k \partial x_k}} + \overline{u_i f_i} \tag{18}$$

Simplifying the above equation term-wise, keeping in mind that the turbulence reaches a steady state, terms involving gradients of averaged quantities will be zero (due to homogeniety), and that $u_{k,k} = 0$ (continuity), we can obtain the following simplified equation:

$$\overline{f_i u_i} - \nu\overline{u_{i,k}u_{i,k}} \quad = \quad 0 \tag{19}$$

The first term is the rate of production of energy (P), due to the spoon, and the second term is the rate of dissipation (ϵ). Production is occuring only in the large scales (from given conditions). Interestingly, dissipation occurs purely due to viscosity, and is always taking away energy from the flow. At what scales do dissipation occur ? Let's do a scale decomposition of ϵ, by first noting (from Eqn (12)):

$$\frac{\partial u_i}{\partial x_j}(\mathbf{x}) = \sum_{\boldsymbol{\kappa}} ik_j \hat{u}_i(\boldsymbol{\kappa}) \exp[i\kappa \cdot \mathbf{x}] \tag{20}$$

which leads to

$$\begin{aligned} \nu \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}}(\mathbf{x}) &= \sum_{\boldsymbol{\kappa}} \overline{-ik_j \hat{u}_i(\boldsymbol{\kappa}) ik_j \hat{u}_i(-\boldsymbol{\kappa})} \\ &= \sum_{\boldsymbol{\kappa}} k^2 \overline{\hat{u}_i(\boldsymbol{\kappa}) \hat{u}_i(-\boldsymbol{\kappa})} \end{aligned} \tag{21}$$

Again, we would like to express ϵ as an integral over a smooth spectra, and therefore we define:

$$\hat{\epsilon}(\kappa) = \sum_{\boldsymbol{\kappa'}} \kappa'^2 \overline{\hat{u}_i(\boldsymbol{\kappa'}) \hat{u}_i^*(\boldsymbol{\kappa'})} \delta(\kappa - |\boldsymbol{\kappa'}|) \tag{22}$$

so that $\epsilon = \int_0^\infty \hat{\epsilon}(\kappa) \mathrm{d}\kappa$. We can in fact take κ'^2 out of the summation (because $\delta(\kappa - |\boldsymbol{\kappa'}|)$ is nonzero over only the $\kappa' = \kappa$ shell) and obtain:

$$\hat{\epsilon}(\kappa) = \kappa^2 \sum_{\boldsymbol{\kappa'}} \overline{\hat{u}_i(\boldsymbol{\kappa'}) \hat{u}_i^*(\boldsymbol{\kappa'})} \delta(\kappa - |\boldsymbol{\kappa'}|) = \kappa^2 E(\kappa) \tag{23}$$

Assuming $E(\kappa) \sim k^{-n}$ the condition $n < 2$ would imply that $\hat{\epsilon}(\kappa)$ increases monotonically with κ; i.e. dissipation occurs mainly in the small scales.

The above discussion yields the following conclusions. Production of energy $P(= \epsilon)$ occurs in the large scales. Dissipation of energy (ϵ) occurs in the small scales. There is an intermediate range, which does not primarily produce or dissipate energy. Rather, it transfers energy from the large scales to the small scales. To obtain an estimate of the small dissipative scales, we note that the only important quantities in these scales are ϵ and ν. Now, $[\epsilon] = L^2T^{-3}$ and $[\nu] = L^2T^{-1}$. Therefore, the only length-scale we can obtain is $\eta = (\nu^3/\epsilon)^{1/4}$. Similarly, the relevant quantities for the large scale are K and ϵ. Noting that $[K] = L^2T^{-2}$, we obtain the "integral length scale", or size of the largest eddies, as $L = K^3/\epsilon$.

On the basis of dimensional analysis, we can now derive expressions for energy spectra $E(\kappa)$ and $S_2(r)$ for the "inertial range" in κ or r. Note that both $E(\kappa)$ and $S_2(r)$ yield scale-specific information. There exists a range of scales given by $\eta \ll r \ll L$ or, equivalently, $2\pi/L \ll \kappa \ll 2\pi/\eta$, which can be said to be an overlap region between the integral scales and the dissipative scales. These are the scales where only energy transfer is taking place. These scales are too small to be influenced by K or L and too large to be influenced by ν. The only relevant quantities here is the energy transfer rate, $\Pi = \epsilon$, and the length scale r (or $2\pi/\kappa$). Thus, noting that $[E] = L^3T - 2$, $[\epsilon] = L^2T^{-3}$ and $[\kappa] = L^-1$, we obtain, after postulating that $E(\kappa) = F_E(\epsilon, \kappa)$ and using dimensional analysis:

$$E(\kappa) = C_\kappa \epsilon^{2/3} \kappa^{-5/3} \tag{24}$$

where C_κ is a universal constant, also known as the "Kolmogorov" constant. Similarly, if we postulate that $S_2(r) = F_S(\epsilon, r)$, and if we observe that $[S_2] = L^2T^{-2}$, then dimensional analysis yields:

$$S(r) = C\epsilon^{2/3}r^{2/3} \tag{25}$$

where C is again a universal constant. The exponents $-5/3$ and $2/3$ are consistent with Eqn (15)–(16) discussed earlier. That is if $E(\kappa) \sim \kappa^{-n} \sim \kappa^{-5/3}$, then it follows that $n = 5/3$, so that $S_2(r) \sim r^{n-1} \sim r^{2/3}$.

We now discuss time-scales for the different scales. Let us say that we measure the velocity scale of an "eddy" of size l by taking the difference in velocity $v_l \sim |\mathbf{u}(\mathbf{x} + l\mathbf{i}) - \mathbf{u}(\mathbf{x})|$. This eddy, having a round shape initially, elongates, due to its own flow field, and eventually breaks up into two smaller eddies. We are basically describing a hypothetical mechanism for the Kolmogorov cascade here. The time scale needed for the eddy to break is clearly around $t_l \sim l/v_l$. The energy transfer rate to the small scales is given by $\Pi_l = v_l^2/t^l = v_l^3/l$. Noting that $\Pi_l \sim \epsilon$, we then obtain $v_l^3 \sim \epsilon l$, or $v_l \sim \epsilon^{1/3}l^{1/3}$. Note that this is consistent with $v_r^2 \sim S_2(r) \sim \epsilon^{2/3}r^{2/3}$. Moreover, we obtain the time-scale for an eddy of size l to break up (or "eddy turnover time") as $t_l \sim \epsilon^{-1/3}l^{2/3}$. Clearly, as the eddies become smaller, they break more quickly. This fact forms the basis for the "local isotropy" condition mentioned earlier. The large, anisotropic eddies break up very slowly, while the smaller scales, turning over much faster, can quickly adjust themselves and attain an isotropic cascade. We can also obtain an independent estimate of the dissipative scales η using this argument, via the definition of a viscous time scale $t_{diff} = l^2/\nu$, which is the time taken for an eddy of length l to simply diffuse out due to viscosity. If $t_l < t_{diff}$, then the eddy will break up due to inertial forces, otherwise it will break up due to viscosity. At $t_l = t_{diff}$, the threshold for viscous breakup is reached, and we obtain $l = \eta = \nu^3/\epsilon$. The quantity η is called the "Kolmogorov scale", and is the smallest length scale in isotropic turbulence.

Using the above estimates of the range of length and time scales, we can also derive the computational cost for simulating isotropic turbulence. The largest scales in the flow is the integral length scale, $L = K^{3/2}/\epsilon$, while the smallest scale is $\eta = (\nu^3/\epsilon)^{1/4}$. Clearly, the number of grid points needed is then $N = (L/\eta)^3 = \mathrm{Re}_L{}^{9/4}$, where $\mathrm{Re}_L = K^2/(\nu\epsilon)$ is known as the turbulent Reynolds number. The simulation will need to be run for at least one eddy-turnover time, $t_L = K/\epsilon$. The time stepping will be limited by the Courant number, so that $\Delta t \sim \eta/K^{1/2}$. Hence, number of time steps in the simulation will be $N_t = t_L/\Delta t = \mathrm{Re}_L^{3/4}$. Hence, the net computational cost needed is $NN_t \sim \mathrm{Re}_L^3$. Clearly, increasing the Reynolds number leads to a steep increase in computational expense.