ME724:Reynolds Stress Transport Models

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April 1, 2016

Introduction

- 2-Eqn models ($k - \epsilon$, $k - \omega$) use eddy viscosity hypothesis:

$$R_{ij} = \frac{2}{3} K \delta_{ij} - 2\nu_t S_{ij} \tag{1}$$

  Issues:
  - Reynolds stress responds instantaneously to local strain rate
  - Anisotropy b_{ij} is proportional to S_{ij} (Algebraic dependence)
  - Normal componeents of R_{ij} always equal ($R_{11} = R_{22} = R_{33}$) if S_{ij} has only non-diagonal components (e.g. channel flow)
  - Insensitive to rotation rate
- **Reynolds Stress Transport** (RST) models: Evolve all six components of R_{ij}
  - Use the exact equations to construct the model
  - Typically, k and ϵ equations are solved as well

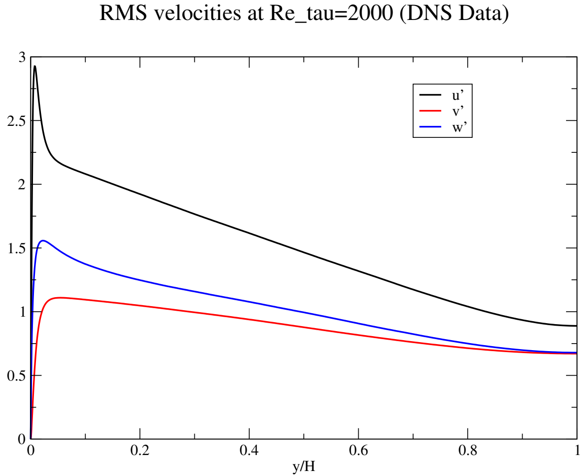
Exact RST Equations

$$\frac{DR_{ij}}{Dt} = -\phi_{ij} - \epsilon_{ij} + P_{ij} + \nu\nabla^2 R_{ij} - \frac{\partial \Gamma_{kij}}{\partial x_k}$$

$$\begin{aligned}
\phi_{ij} &= \langle u_j p_{,i} \rangle + \langle u_i p_{,j} \rangle \quad \text{Pressure Redistribution} \\
\epsilon_{ij} &= 2\nu \left\langle u_{i,k} u_{j,k} \right\rangle \quad \text{Dissipation rate} \\
P_{ij} &= -R_{jk} U_{i,k} - R_{ik} U_{j,k} \quad \text{Production} \\
\Gamma_{kij} &= \langle u_k u_i u_j \rangle \quad \text{Turbulent Flux} \\
\frac{\partial \Gamma_{kij}}{\partial x_k} &: \quad \text{Turbulent Transport} \quad \nu\nabla^2 R_{ij} : \text{Viscous Transport}
\end{aligned}$$

Need to model: Pressure Redistribution, Dissipation Rate and Turbulent Transport

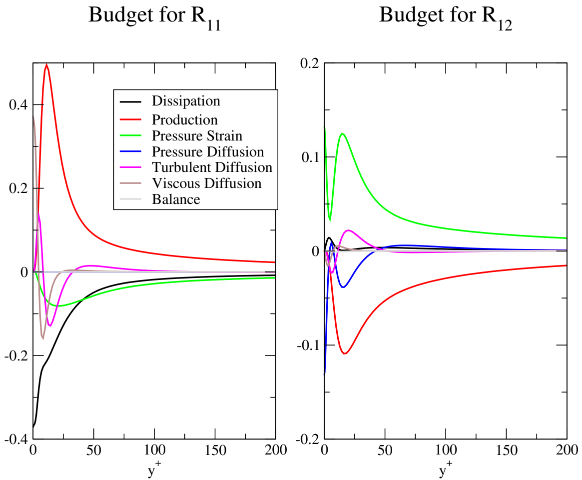
Data from DNS at $Re_\tau = 2000$ (Jimenez and Hoyas, JFM, 2006)



R_{ij} is highly anistropic near the wall. R_{11} is the strongest component, because $P_{22} = P_{33} = 0$.

Budget Terms for Turbulent Channel Flow

Data from DNS at $Re_\tau = 2000$ (Jimenez and Hoyas, JFM, 2006)



Pressure redistribution (=pressure strain+pressure transport), along with Production and Dissipation, is very important

But why do we care about normal components, since only R_{12} is important in Channel Flow ?
Because normal components can influence budget terms for R_{12}.
Let's examine P_{12} if $U = U(x_2)$ only:

$$P_{12} = -R_{1k}U_{2,k} - R_{2k}U_{1,k} = -R_{22}U_{1,2} \tag{2}$$

Thus, production of Reynolds shear stress depends on variance of wall normal velocity fluctuations, R_{22}. But $P_{22} = 0$, so where does it get energy from ?

Pressure Redistribution

Note that:

$$\begin{aligned}\phi_{ij} = \langle u_j p_{,i}\rangle + \langle u_i p_{,j}\rangle &= \overbrace{-\langle p(u_{i,j} + u_{j,i})\rangle}^{\Pi_{ij}} + \frac{\partial T_{kij}}{\partial x_k} \\ \text{where} \quad T_{kij} &= \delta_{ki}\langle u_j p\rangle + \delta_{kj}\langle u_i p\rangle\end{aligned}$$

For incompressible flows, $\Pi_{ii} = 0$, implying:

$$\phi_{ii} = \frac{\partial T_{kii}}{\partial x_k} = 2\frac{\partial \langle u_k p\rangle}{\partial x_k}$$

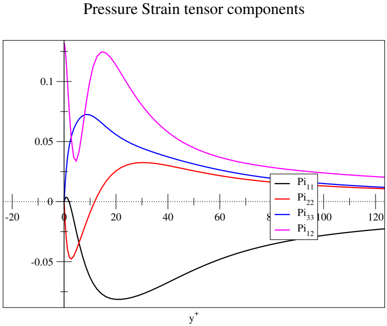
For homogeneous flows, $\phi_{ii} = 0$; Pressure redistribution does not provide any net kinetic energy.

For inhomogeneous, $\frac{\partial T_{kij}}{\partial x_k}$ can be neglected, so that $\phi_{ij} \approx \Pi_{ij}$

Modeling ϕ_{ij} complicated due to non-local nature of pressure.

Pressure Strain

$\Pi = -\langle p(u_{i,j} + u_{j,i}) \rangle$ is known as the "pressure strain"
Data from DNS at $Re_\tau = 2000$ (Jimenez and Hoyas, JFM, 2006)



Presure strain transfers energy from R_{11} to R_{22} and R_{33}

Turbulent Transport

Start with usual gradient diffusion hypothesis:

$$T_{kij} = \langle u_k u_i u_j \rangle \quad = \quad -C_s \frac{K^2}{\epsilon} \frac{\partial \langle u_i u_j \rangle}{\partial x_k}$$

Near the wall $\langle u_2 u_2 \rangle$ is more important for transport in wall-normal direction. So a better model is:

$$\langle u_k u_i u_j \rangle \quad = \quad -C_s \frac{K}{\epsilon} \langle u_k u_l \rangle \frac{\partial \langle u_i u_j \rangle}{\partial x_l}$$

Check: Near the wall, $\frac{\partial T_{2ij}}{\partial x_2}$ will dominate, for which:

$$\frac{\partial \langle u_2 u_i u_j \rangle}{\partial x_2} \quad = \quad -C_s \frac{K}{\epsilon} \langle u_2 u_2 \rangle \frac{\partial \langle u_i u_j \rangle}{\partial y} \qquad (3)$$

Dissipation Rate Tensor

Local isotropy of small scales implies that dissipation rate will be almost isotropic outside the viscous layer, i.e:

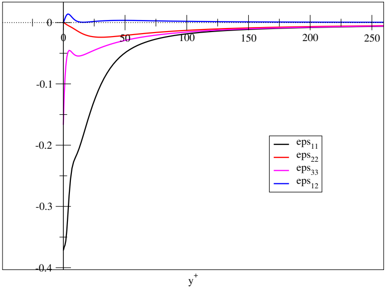
$$\epsilon_{ij} = 2\nu \left\langle u_{i,k} u_{j,k} \right\rangle \quad = \quad \frac{2}{3}\delta_{ij}\epsilon$$

(Note that $\epsilon = \nu \langle u_{i,k} u_{i,k} \rangle = \frac{1}{2}\epsilon_{ii}$)

Dissipation Rate Tensor

Data from DNS at $Re_\tau = 2000$ (Jimenez and Hoyas, JFM, 2006)

Dissipation rate tensor components



ϵ_{ij} is indeed isotropic, except very close to the wall

Let's lump in the anisotropic part of ϵ_{ij} into the redistribution:

$$\Phi_{ij} = -(\phi_{ij} - \frac{1}{3}\delta_{ij}\phi_{kk} + \epsilon_{ij} - \frac{2}{3}\epsilon\delta_{ij})$$

So that, for homogeneous turbulence

$$\frac{\partial R_{ij}}{\partial t} = P_{ij} + \Phi_{ij} - \frac{2}{3}\epsilon\delta_{ij}$$

Note that $\phi_{kk} = 0$ for homogeneous turbulence.
We need to representation for Φ_{ij}:

$$\Phi_{ij} = F_{ij}(R_{ij}, U_{i,j}, \epsilon, \delta_{ij}) \tag{4}$$

But let's first further understand the nature of pressure redistribution..

Pressure Redistribution

Poisson Eqn for pressure fluctuation is:

$$\nabla^2 p \quad = \quad \overbrace{-2U_{i,j}u_{j,i}}^{\text{"Rapid"}} - \overbrace{(u_i u_j - R_{ij})_{,ij}}^{\text{"Slow"}}$$

Redistribution term needs to be separated into "rapid" and "slow" terms:

$$\Phi_{ij} \quad = \quad \Phi^{(r)}[U_{i,j}, R_{ij}, \epsilon, \delta_{ij}] + \Phi_{ij}^{(s)}[R_{ij}, \epsilon, \delta_{ij}]$$

Also, from dimensional analysis:

$$\Phi_{ij} = \epsilon \mathcal{F}_{ij}[b_{ij}, \frac{K}{\epsilon}, U_{i,j}, \delta_{ij}]$$

where $b_{ij} = \frac{R_{ij}}{K} - \frac{2}{3}\delta_{ij}$

"Slow" Pressure Redistribution

Φ_{ij} is responsible for transferring energy between components (i.e. it controls anisotropy), and does not contribute to K
We will therefore try to look at equation for b_{ij}:

$$\begin{aligned}
\frac{db_{ij}}{dt} &= \frac{\partial(R_{ij}/K)}{\partial t} = \frac{1}{K}\frac{\partial R_{ij}}{\partial t} - \frac{R_{ij}}{K^2}\frac{\partial K}{\partial t} \\
&= \frac{1}{K}[P_{ij} + \Phi_{ij} - \frac{2}{3}\epsilon\delta_{ij}] - \frac{R_{ij}}{K^2}[P - \epsilon] \\
&= -b_{ik}U_{j,k} - b_{jk}U_{i,k} - \frac{4}{3}S_{ij} - \left(b_{ij} + \frac{2}{3}\delta_{ij}\right)\frac{P}{K} \\
&\quad + b_{ij}\frac{\epsilon}{K} + \frac{\epsilon}{K}\mathcal{F}_{ij}
\end{aligned}$$

In the absence of mean velocity gradients:

$$\frac{db_{ij}}{dt} = b_{ij}\frac{\epsilon}{K} + \frac{\epsilon}{K}\mathcal{F}^{(s)}_{ij}\left[b_{ij}, \frac{K}{\epsilon}, \delta_{ij}\right]$$

"Slow" Pressure Redistribution

According to experiments, if $U_{i,j} = 0$, anisotropic turbulence "returns" to isotropic state over some time scale, i.e. $b_{ij} \to 0$ in the absence of any velocity gradient.

Rotta's model:

$$\begin{aligned}
\mathcal{F}_{ij}^{(s)} &= -C_1 b_{ij} \\
\Rightarrow \Phi_{ij}^{(s)} &= -C_1 \epsilon b_{ij} \\
\Rightarrow \frac{db_{ij}}{dt} &= (1 - C_1)\frac{b_{ij}}{T}
\end{aligned}$$

where $T = K/\epsilon$. $C_1 = 1.5 \to 2.0$ is typically used.
$C_1 > 0$ needed for stable equilibrium.
Nonlinear models (in b_{ij}) also possible, but first we need to understand some tensor representation theory...

Representation Theory of Tensors

Question: Say we are trying to model a tensor as $\phi_{ij} = F_{ij}[\mathbf{A}, \mathbf{B}, \mathbf{C}...]$, where $A_{ij}, B_{ij}, C_{ij}, ..$ are also tensors. What is a tensorially consistent form for $F_{ij}[\cdot]$?
Recall, for a *vector* $\mathbf{x}$, if the coordinate system is represented by basis $\bar{\mathbf{e}}_i$ instead of $\mathbf{e}_i$, where $a_{ij} = \mathbf{e}_i \cdot \bar{\mathbf{e}}_j$ then:

$$\begin{aligned} \mathbf{x} &= \bar{\mathbf{e}}_i \bar{x}_i = \mathbf{e}_j x_j \\ \Rightarrow \bar{x}_i &= \bar{\mathbf{e}}_i \cdot \mathbf{e}_j x_j = a_{ji} x_i \end{aligned}$$

Vector is a "first order tensor", because it has only one subscript index. For a second order tensor, $\mathbf{b}$:

$$\begin{aligned} \mathbf{b} &= \bar{\mathbf{e}}_k \bar{\mathbf{e}}_l \bar{b}_{kl} = \mathbf{e}_i \mathbf{e}_j b_{ij} \\ \Rightarrow \bar{b}_{kl} &= a_{ik} a_{jl} b_{ij} \end{aligned}$$

Why do we care about tensors ? Because R_{ij} (and therefore b_{ij}) are tensors.

Proof: First note that u_i is a tensor, so that $\bar{u}_k = a_{ki} u_i$

$$
\begin{aligned}
R_{ij} &= \langle u_i u_j \rangle \\
\Rightarrow \bar{R}_{kl} &= \langle \bar{u}_k \bar{u}_l \rangle = a_{ki} a_{lj} \langle u_i u_k \rangle \\
\Rightarrow \bar{R}_{kl} &= a_{ki} a_{lj} R_{ij}
\end{aligned}
$$

An *Isotropic Cartesian Tensor Function* $F_{ij}[\mathbf{A},\mathbf{B},\mathbf{C}...]$ has the property that after a coordinate transformation:

$$F_{ij}[\bar{\mathbf{A}},\bar{\mathbf{B}},\bar{\mathbf{C}}...] \quad = \quad a_{ki}a_{lj}F_{ij}[\mathbf{A},\mathbf{B},\mathbf{C}...]$$

Example: Say, $F_{ij}[\mathbf{A},\mathbf{B}] = A_{ik}B_{kj}$, then

$$\begin{aligned} F_{ij}[\bar{\mathbf{A}},\bar{\mathbf{B}}] = \bar{A}_{ik}B_{kj} &= a_{li}a_{mk}a_{qk}a_{sj}A_{lm}B_{qs} \\ &= a_{li}\delta_{mq}a_{sj}A_{lm}B_{qs} = a_{li}a_{sj}A_{lq}B_{qs} \\ &= a_{li}a_{sj}F_{ls}[\mathbf{A},\mathbf{B}] \end{aligned}$$

Clearly, tensor products $\mathbf{A}\cdot\mathbf{B}$ are Isotropic Cartesian Tensor Functions. Is there a general way to construct such functions ?

Method to Construct Isotropic Cartesian Tensor Functions:
Take any two vectors $\mathbf{a}$, $\mathbf{b}$. Clearly, $a_i b_j F_{ij}(\mathbf{A}, \mathbf{B}, ..)$ will be :

- A scalar quantity (i.e. does not change with rotation)
- Containing terms linear in $\mathbf{a}$ and $\mathbf{b}$ (e.g. something like $a_i b_j F_{ij}(\mathbf{A}) = \frac{1}{a_i b_j A_{ij}}$ not possible).
- Containing terms that can depend on $\mathbf{A}$, $\mathbf{B}$..

For example, we can just list all the possible terms in $a_i b_j F_{ij}(\mathbf{A})$..

$$\begin{aligned} a_i b_j F_{ij} &= C_1 a_k b_k + C_2 a_k b_l A_{kl} + C_3 a_k b_m A_{kl} A_{lm} + \\ &\quad C_4 a_k A_{kl} A_{lm} A_{mn} b_n + ... \end{aligned}$$

Here $C_1, C_2, ..$ are functions of scalar invariants of $\mathbf{A}$. So, most general form for F_{ij} will be:

$$F_{ij} = C_1 \delta_{ij} + C_2 A_{ij} + C_3 A_{ik} A_{kj} + C_4 A_{ik} A_{kl} A_{lj} + ..$$

Representation Theory of Tensors

So, the most general form for $\mathbf{F}(\mathbf{A})$ seems to be:

$$\mathbf{F} = C_1\mathbf{I} + C_2\mathbf{A} + C_3\mathbf{A}^2 + C_4\mathbf{A}^3 + ..$$

Which can have infinite number of terms ! Thankfully, Cayley Hamilton Theorem says :

$$\begin{aligned} A^3_{ij} &= III\delta_{ij} - IIA_{ij} + IA^2_{ij} \\ I &= \lambda_1 + \lambda_2 + \lambda_3 = A_{kk} \\ II &= -\frac{1}{2}[A^2_{kk} - I^2] \\ III &= \frac{1}{3}[A^3_{kk} + III - A^2_{kk}I] \end{aligned}$$

Therefore, most general representation is :

$$\mathbf{F} = C_1\mathbf{I} + C_2\mathbf{A} + C_3\mathbf{A}^2$$

where $C_\alpha = C_\alpha(I, II, III)$

Nonlinear Terms In Slow Pressure Strain

More general model for slow pressure strain is

$$\mathcal{F}_{ij}^{(s)} = C_0 \delta_{ij} - C_1 b_{ij} + C_1^n b_{ij}^2$$

$\mathcal{F}_{ii}^{(s)} = 0$ implies $C_0 = -\frac{C_1^n}{3} b_{kk}^2$ Thus:

$$\begin{aligned} \frac{db_{ij}}{dt} &= b_{ij}\frac{\epsilon}{K} + \frac{\epsilon}{K}\mathcal{F}_{ij}^{(s)}\left[b_{ij}, \frac{K}{\epsilon}, \delta_{ij}\right] \\ &= (1 - C_1)\frac{b_{ij}}{T} + C_1^n \frac{(b_{ij}^2 - \frac{1}{3} b_{kk}^2 \delta_{ij})}{T} \end{aligned}$$

"Realizability" is still a major concern, i.e. at all times $\langle R_{\alpha\alpha} \rangle > 0$ for $\alpha = 1, 2, 3$

Realizability for Slow Pressure Strain

We can analyze evolution of b_{ij} in the principle coordinate frame, where diagonal terms of b_{ij} are zero.
Let's look at $b_{11} = R_{11}/K - 2/3$. $R_{11} > 0$ implies $b_{11} > -2/3$
At $b_{11} = -2/3$, $b_{33} = -b_{11} - b_{22} = 2/3 - b_{22}$. We can write $\frac{db_{11}}{dt}$ as:

$$T\frac{db_{11}}{dt} \quad = \quad -\frac{2}{3}(1 - C_1) + C_1^n(b_{22}^2 - \frac{2}{3}b_{22} - \frac{2}{9})$$

RHS has minimum value of $\frac{2}{3}[C_1 - 1 - \frac{2}{3}C_1^n]$. For $\frac{db_{11}}{dt} > 0$ at this minimum RHS value, we therefore need:

$$C_1^n \quad < \quad \frac{3}{2}(C_1 - 1)$$

Typical values : $C_1 = 1.7$, $C_1^n = 1.0$

"Rapid" Pressure Strain

Recall:

$$\phi_{ij} = \langle u_j p_{,i} \rangle + \langle u_i p_{,j} \rangle$$

Focusing on the pressure fluctuation due to homogeneous strain rate $U_{i,j}$:

$$\nabla^2 p = -2 U_{k,l} u_{l,k}$$

Solving for p..

$$\begin{aligned}
p(\mathbf{x}) &= (\nabla^2)^{-1} \left[-2 U_{k,l} u_{l,k} \right] \\
&= \frac{U_{k,l}}{2\pi} \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} \frac{\partial u_l(\mathbf{x}')}{\partial x'_k} d^3\mathbf{x}' \\
\Rightarrow \frac{\partial p(\mathbf{x})}{\partial x_i} &= \frac{U_{k,l}}{2\pi} \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} \frac{\partial^2 u_l(\mathbf{x}')}{\partial x'_i \partial x'_k} d^3\mathbf{x}' \\
&= -U_{k,l} (\nabla^2)^{-1} [2 u_{l,ki}]
\end{aligned}$$

"Rapid" Pressure Strain

After making a change of variable $\xi_i = x'_i - x_i$ and some algebra, we will get:

$$
\begin{aligned}
\phi_{ij}^{(r)} &= -M_{ijkl}U_{k,l} \\
\text{where } M_{ijkl} &= 2(\nabla_{\xi}^{2})^{-1}\left[R_{jl,ik}\left(\vec{\xi}\right) + R_{il,jk}\left(\vec{\xi}\right)\right]
\end{aligned}
$$

We can derive the following for M_{ijkl}:

- Contracting j, k, we get $M_{ijjl} = 2R_{il}$ (Constraint 1)
- $\phi_{ii} = 0$ implies $M_{iikl} = 0$ (Constraint 2)
- $\phi_{ij} = \phi_{ji}$ implies $M_{ijkl} = M_{jikl}$ (Constraint 3)

Also, clearly, $M_{ijkl} = M_{ijkl}[\delta_{ij}, b_{ij}]$ (i.e. does not depend on $U_{i,j}$)

"Rapid" Pressure Strain

An important limiting case is isotropic homogeneous turbulence undergoing constant shear. In this case, at initial time, $|b_{ij}| \approx 0$, and we can approximate $M_{ijkl} = M_{ijkl}[\delta_{ij}]$.
Most general representation for M_{ijkl}:

$$M_{ijkl} = A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk}$$

Constraint 3 implies $B = C$, so that:

$$M_{ijkl} = A\delta_{ij}\delta_{kl} + B[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$$

Constraint 2 implies $3A = -2B$, and Constraint 1 implies:

$$M_{ijjl} = [-\frac{2}{3}B + 4B]\delta_{il} = 2R_{il} = 2K\left[b_{il} + \frac{2}{3}\delta_{il}\right]$$

"Rapid" Pressure Strain

Neglecting b_{ij}, we get:

$$M_{ijkl} = \frac{K}{15}\left[6\delta_{ik}\delta_{jl} + 6\delta_{il}\delta_{jk} - 4\delta_{ij}\delta_{kl}\right]$$

No model constants ! For $\phi_{ij}^{(r)}$, we get:

$$\phi_{ij}^{(r)} = \frac{2}{5}K\left[U_{i,j} + U_{j,i}\right] \qquad (5)$$

The "isotropization of production" model (IP model) for $\phi_{ij}^{(r)}$ proposes, for large b_{ij}:

$$\phi_{ij}^{(r)} = -\frac{3}{5}\left[P_{ij} - \frac{2}{3}\delta_{ij}P\right] \qquad (6)$$

For $b_{ij} = 0$, eqns (5) and (6) are the same.
Basic idea of IP model: ϕ_{ij}^{r} tends to reduce anisotropy in the production, or it tends to isotropize net production.

Total Pressure Redistribution

Final model (using only linear term for $\phi^{(r)}$) for total pressure redistribution is:

$$\begin{aligned}\Phi_{ij} &= -C_1 \epsilon b_{ij} - C_2 \left[P_{ij} - \frac{2}{3}\delta_{ij} P \right] \\ C_1 &= 1.7, \quad C_2 = 3/5\end{aligned}$$

Also known as the Launder-Reece-Rodi Isotropization of Production (LRR-IP) model.

Asymptotic State of Anisotropy

Validate LRR-IP model against DNS data by applying simple shear:

$$\frac{\partial U_i}{\partial x_j} = S\delta_{i1}\delta_{j2}$$

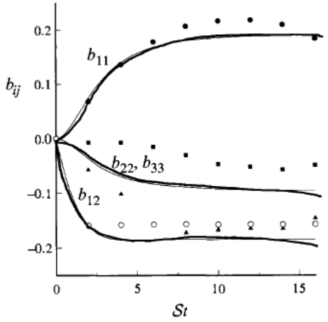


Fig. 11.15. Reynolds-stress anisotropies in homogeneous shear flow. Comparison of LRR-IP model calculations (lines) with the DNS data of Rogers and Moin (1987) (symbols): •, b_{11}; ○, b_{12}; triangles, b_{22}; squares, b_{33}.

Analytical Solutions for Homogeneous Shear Flows

Find $b_{ij}(t)$, starting with isotropic turbulence, $b_{ij}(0) = 0$.
Assumptions:

$$K(t) = K_0 \exp(\lambda t), \quad \epsilon(t) = \epsilon_0 \exp(\lambda t)$$

Hence, $K/\epsilon = K_0/\epsilon_0$ for all time t.
From DNS data: Anisotropy components tend to a constant value: $\lim_{t\to\infty} b_{ij}(t) = b_{ij}^{\infty}$

$$\Rightarrow \lim_{t\to\infty} \frac{P}{\epsilon} = \frac{-K(t) b_{ij}^{\infty} U_{i,j}}{\epsilon_0 \exp(\lambda t)} = -\frac{SK_0}{\epsilon_0} b_{12}^{\infty} = \frac{P^{\infty}}{\epsilon^{\infty}} = \text{constant}$$

Analytical Solutions for Homogeneous Shear Flows

In the limit $t \to \infty$, using the IP model and Rotta model for Φ_{ij}:

$$\begin{aligned}
\frac{db_{ij}}{dt} = 0 &= \frac{1}{K}\left[P_{ij} + \Phi_{ij} - \frac{2}{3}\epsilon\delta_{ij}\right] - \frac{1}{K}\left[b_{ij} + \frac{2}{3}\delta_{ij}\right][P - \epsilon] \\
\Rightarrow 0 &= P_{ij} - C_2\left[P_{ij} - \frac{2}{3}\delta_{ij}P\right] - C_1\epsilon b_{ij} - \frac{2}{3}\delta_{ij}\epsilon - \\
& \quad b_{ij}(P - \epsilon) - \frac{2}{3}\delta_{ij}(P - \epsilon) \\
\Rightarrow b_{ij} &= \frac{1}{2}\Theta\left(P_{ij} - \frac{2}{3}P\delta_{ij}\right)/P, \quad \text{where} \quad \Theta = \frac{(1 - C_2)P/\epsilon}{C_1 - 1 + P/\epsilon}
\end{aligned}$$

For this flow, $P_{11} = -R_{12}S = -Kb_{12}S = 2P$, $P_{22} = P_{33} = 0$, implying:

$$\begin{aligned}
b_{11} &= \frac{2}{3}\Theta, \quad b_{22} = b_{33} = -\frac{\Theta}{3} \\
b_{12} &= -\sqrt{\frac{1}{6}\Theta(1 - \Theta)}
\end{aligned}$$

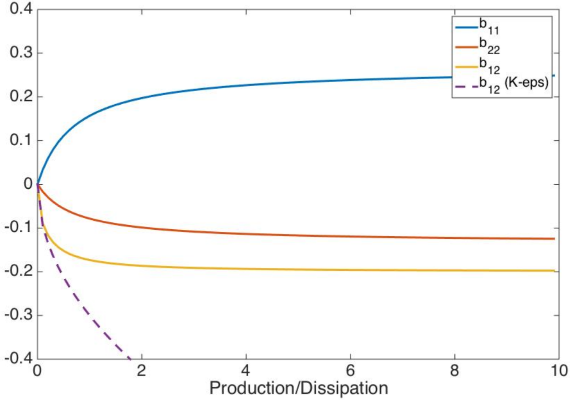
Analytical Solutions for Homogeneous Shear Flows

For $P/\epsilon \to \infty$, $\Theta \to (1 - C_2)$, b_{ij} goes to a constant value.
What does $K - \epsilon$ model predict ? For any value of P/ϵ, $b_{11} = b_{22} = b_{33} = 0$.
Also according to 2 eqn models:

$$\begin{aligned} b_{12} &= -C_\mu (SK/\epsilon) \\ \frac{P}{\epsilon} &= -R_{12}S = -b_{12}\frac{SK}{\epsilon} \\ &= \frac{b_{12}^2}{C_\mu} \\ \Rightarrow b_{12} &= \pm\sqrt{C_\mu \frac{P}{\epsilon}} \end{aligned}$$

So, when $\frac{P}{\epsilon} \to \infty$, $b_{12} \to \pm\infty$, which is incorrect.

Analytical Solutions for Homogeneous Shear Flows



Experimental results for $P/\epsilon = 1.6$ (Tavaularis and Karnik, 1989) :
$b_{11} = 0.36 \pm 0.08$, $b_{22} = -0.22 \pm 0.05$, $b_{33} = -0.14 \pm 0.06$,
$b_{12} = -0.32 \pm 0.02$

Modeling the Effects of Inhomogeniety

Dissipation rate $\epsilon_{ij} = 2\nu \langle u_{i,k} u_{j,k} \rangle$ is non-zero and anisotropic at wall. Important to capture its wall asymptotics. Recall, near the wall:

$$
\begin{aligned}
u_1(\mathbf{x}, t) &= a_1(t) y + O(y^2), \quad u_2(\mathbf{x}, t) = a_2(t) y^2 + O(y^3), \\
u_3(\mathbf{x}, t) &= a_3(t) y + O(y^2)
\end{aligned}
$$

Implies, that, near $y = 0$:

$$
\begin{aligned}
K(y) &= \frac{1}{2} \left\langle u_1^2 + u_2^2 + u_3^2 \right\rangle = \frac{1}{2} \left\langle a_1^2 + a_3^2 \right\rangle y^2 + O(y^4) \\
\epsilon(y) &= \frac{1}{2} \epsilon_{kk} = \nu \left\langle u_{1,2}^2 + u_{2,2}^2 + u_{3,2}^2 \right\rangle = \nu \left\langle a_1^2 + a_3^2 \right\rangle + O(y^2)
\end{aligned}
$$

Modeling the Effects of Inhomogeneity

Also,

$$
\begin{aligned}
R_{ij} &= \begin{bmatrix} \left\langle a_1^2 \right\rangle y^2 & \left\langle a_1 a_2 \right\rangle y^3 & \left\langle a_1 a_3 \right\rangle y^2 \\ \left\langle a_2 a_1 \right\rangle y^3 & \left\langle a_2 a_2 \right\rangle y^4 & \left\langle a_2 a_3 \right\rangle y^3 \\ \left\langle a_3 a_1 \right\rangle y^2 & \left\langle a_3 a_2 \right\rangle y^3 & \left\langle a_3 a_3 \right\rangle y^2 \end{bmatrix} + \text{Higher Order Terms} \\
\epsilon_{ij} &= 2\nu \begin{bmatrix} \left\langle a_1^2 \right\rangle & 2\left\langle a_1 a_2 \right\rangle y & \left\langle a_1 a_3 \right\rangle \\ 2\left\langle a_2 a_1 \right\rangle y & 4\left\langle a_2^2 \right\rangle y^2 & 2\left\langle a_2 a_3 \right\rangle y \\ 2\left\langle a_3 a_1 \right\rangle & 2\left\langle a_3 a_2 \right\rangle y & \left\langle a_3^2 \right\rangle \end{bmatrix} + \text{Higher Order Terr}
\end{aligned}
$$

Modeling the Effects of Inhomogeniety

Possible to show, that near the wall:

$$\frac{\epsilon_{ij}}{\epsilon} = \begin{cases} \frac{R_{ij}}{K} & \text{for} \quad i \neq 2,\ AND\, j \neq 2 \\ 2\frac{R_{ij}}{K} & \text{for} \quad i = 2,\ AND\, j \neq 2 \\ 2\frac{R_{ij}}{K} & \text{for} \quad j = 2,\ AND\, i \neq 2 \\ 4\frac{R_{ij}}{K} & \text{for} \quad i = 2,\ AND\, j = 2 \end{cases}$$

Rotta (1951) proposed following model for dissipation for inhomogeneous flows:

$$\epsilon_{ij} = \frac{R_{ij}}{K}\epsilon$$

Underestimates ϵ_{12} and ϵ_{22} by factors of 2 and 4. Goes to $\epsilon_{ij} = (2/3)\delta_{ij}\epsilon$ in isotropy.

Modeling the Effects of Inhomogeniety

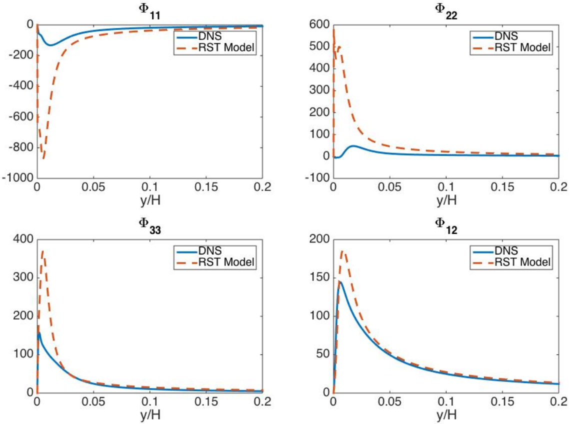
Redistribution tensor is redefined as:

$$\Phi_{ij} = -(\phi_{ij} - \frac{1}{3}\delta_{ij}\phi_{kk} + \epsilon_{ij} - \frac{R_{ij}}{K}\epsilon)$$

Complete RST equation for inhomogeneous flows is then:

$$\frac{DR_{ij}}{Dt} = P_{ij} + \Phi_{ij} - \frac{R_{ij}}{K}\epsilon + \nu\frac{\partial^2 R_{ij}}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k}\left[C_s \frac{K}{\epsilon} R_{kl} \frac{\partial R_{ij}}{\partial x_l}\right]$$

Comparison of Φ_{ij} (redistribution term) from LRR-IP model with DNS data of channel flow at $Re_\tau = 2000$ (Hoyas and Jimenez, 2006)



Elliptic Relaxation (Durbin)

Near wall behavior of LRR-IP model is poor. Non-locatlity of pressure is not accounted for.

Elliptic Relaxation technique attempts to account for non-locality of pressure.

First, note that, in the presence of a wall at $y = 0$:

$$p(\mathbf{x}) = \frac{1}{2\pi} \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} U_{k,l} \frac{\partial u_l(x', z', |y'|)}{\partial x'_k} d^3\mathbf{x}'$$

We can write:

$$\langle u_i p_{,j} \rangle = \frac{1}{4\pi} \int \frac{\left\langle u_i(\mathbf{x}) \partial S(\mathbf{x}') / \partial x'_j \right\rangle}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'$$

Exact form of $S(\mathbf{x})$ is not very crucial here.

Elliptic Relaxation

When turbulence is inhomogeneous, we can model the 2-point correlation in integrand as:

$$\left\langle u_i(\mathbf{x}) \frac{\partial S(\mathbf{x}')}{\partial x'_j} \right\rangle = Q_{ij}(\mathbf{x}') \exp[-|\mathbf{x} - \mathbf{x}'|/L]$$

where L is the length scale of the correlation. Thus:

$$\langle u_i p_{,j} \rangle (\mathbf{x}) = \int Q_{ij}(\mathbf{x}') \frac{\exp[-|\mathbf{x} - \mathbf{x}'|/L]}{4\pi|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \qquad (7)$$

Turns out that this is the solution to the modified Helmholtz equation !

$$\nabla^2 \langle u_i p_{,j} \rangle - \frac{\langle u_i p_{,j} \rangle}{L^2} = -Q_{ij}$$

For complex geometries, solving for above equation is easier than carrying out the integral in eqn (7).

Elliptic Relaxation

Note that, for homogeneous flows, $\frac{\langle u_i p_{,j} \rangle}{L^2} = Q_{ij}$. So, Q_{ij} is simply the homogeneous limit of the redistribution tensor.
Denoting the homogeneous model for redistrubution tensor as:

$$\begin{aligned} \Phi_{ij}^{H} &= -C_1 \epsilon b_{ij} - C_2 \left[P_{ij} - \frac{2}{3} \delta_{ij} P \right] \\ C_1 &= 1.7, \quad C_2 = 3/5 \end{aligned}$$

Elliptic Relaxation

Elliptic relaxation then involves solving the following eqn for $f_{ij}(\mathbf{x}) = \Phi_{ij}(\mathbf{x})/K(\mathbf{x})$:

$$
\begin{aligned}
L^2\nabla^2 f_{ij} - f_{ij} &= -\frac{\Phi_{ij}^H}{K} \\
\text{where} \quad L &= \max\left[c_L\frac{K^{3/2}}{\epsilon}, c-\eta\left(\frac{\nu^3}{\epsilon}\right)^{1/4}\right] \\
C_L = 0.2, \qquad & C_\eta = 80
\end{aligned}
$$

$\Phi_{ij} = f_{ij}K$ is then used to get the final redistribution tensor. Solving for f_{ij} (instead of Φ_{ij}) ensures that $\Phi_{ij} = 0$ at the wall.

Elliptic Relaxation: Boundary Condition

Boundary conditions for f_{ij} can be tricky:

$$f_{ij}|_{y=0} = \begin{cases} -5.0 \lim_{y\to 0} \epsilon \frac{R_{22}}{K^2} & \text{for } i = 2, j = 2 \\ 0 & \text{for } (i,j) \in \{(2,1),(2,3),(1,2),(1,3)\} \\ -\frac{f_{22}}{2}\Big|_{y=0} & \text{for } (i,j) \in \{(1,1),(3,3)\} \end{cases}$$

Let's show this for f_{22}

Elliptic Relaxation: Boundary Condition

At the wall, the RST model gives:

$$\begin{aligned} \epsilon \frac{R_{22}}{K} &= K f_{22} + \nu \frac{\partial^2 R_{22}}{\partial y^2} \\ \epsilon &= \nu \frac{\partial^2 K}{\partial y^2} \end{aligned}$$

Near the wall (i.e. $y \to 0$), $K = K_0 y^2$, $R_{22} = R_0 y^4$ and ϵ is a constant. So,

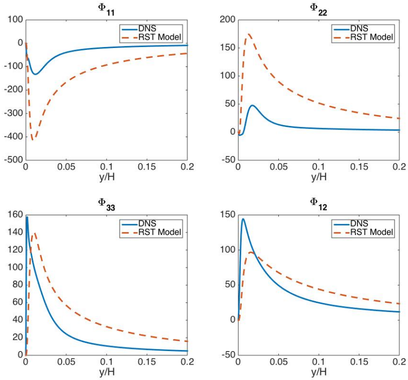
$$\begin{aligned} \epsilon \frac{R_{22}}{K} &= \nu \frac{\partial^2 K}{\partial y^2} \frac{R_{22}}{K} = 2\nu R_0 y^2 \\ \nu \frac{\partial^2 R_{22}}{\partial y^2} &= 12\nu R_0 y^2 = 6\epsilon \frac{R_{22}}{K} \end{aligned}$$

Leading to:

$$f_{22}|_{y=0} = -5 \lim_{y \to 0} \epsilon \frac{R_{22}}{K^2}$$

Elliptic Relaxation

Elliptic relaxation appears to bring down the peaks near the wall, especially for Φ_{22}



The $v^2 - f$ Model (Durbin)

Full RST models with elliptic relaxation can be expensive to implement and may be numerically unstable. Durbin's $v^2 - f$ model is a more inexpensive option.
Only the R_{22} equations are solved like a scalar (along with $k - \epsilon$ model), with elliptic relaxation. R_{22} is renamed as "v^2".

$$\frac{\partial \left\langle v^2 \right\rangle}{\partial t} + U_j \frac{\partial \left\langle v^2 \right\rangle}{\partial x_j} + \epsilon \frac{\left\langle v^2 \right\rangle}{K} = kf + \frac{\partial}{\partial x_k}\left[\nu_T \frac{\partial \left\langle v^2 \right\rangle}{\partial x_k}\right] + \nu \nabla^2 \left\langle v^2 \right\rangle$$

$$L^2 \nabla^2 f - f = -c_2 \frac{P}{K} + \frac{c_1}{T}\left(\frac{\left\langle v^2 \right\rangle}{K} - \frac{2}{3}\right)$$

$$c_2 = 0.3, \quad c_1 = 0.4$$

Boundary condition is: $f = -5 \lim_{y \to 0}\left[\frac{\epsilon \left\langle v^2 \right\rangle}{K^2}\right]$. $\nu_T = C_\mu \left\langle v^2 \right\rangle T$ is used as eddy viscosity in mean velocity equation.

Case Study: Rotating Channel Flow

Non-dimensional Rossby number: $Ro_b = \Omega 2H/U_b$, H = half channel height

Reference for LES data: Piomelli, Ugo, and Junhui Liu. "Large eddy simulation of rotating channel flows using a localized dynamic model." Physics of Fluids 7, no. 4 (1995): 839-848.

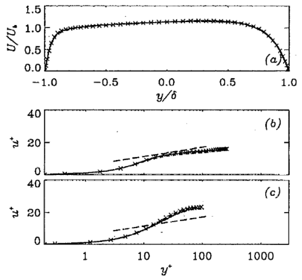


FIG. 1. Mean velocity in the rotating channel. $Re_b=5700$, $Ro_b=0.144$. —— First-order; - - - zeroth order; —·— plane-averaged; – – – 2.5 log $y^+ + 5.0$; × DNS (only every other point is shown). (a) Global coordinates; (b) wall coordinates, unstable side; (c) wall coordinates, stable side.

Physics of Rotating Channel Flow

Positive Ω implies CCW rotation. Mean velocity eqn is:

$$
\begin{aligned}
0 &= -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}(\nu \frac{\partial U}{\partial y} - R_{12}) \\
0 &= -\frac{\partial P}{\partial y} - 2\Omega U - \frac{\partial R_{22}}{\partial y}
\end{aligned}
$$

Coriolis force does not affect mean velocity directly: effect must be through R_{12}. Net production term in DR_{12}/Dt due to Coriolis term is:

$$
-R_{22}\frac{dU}{dy} - 2\Omega(R_{11} - R_{22})
$$

Positive Ω reduces net production of R_{12} if $\frac{dU}{dy}$ is negative, and vice versa. $P = -R_{12}\frac{dU}{dy}$, so TKE production is also reduced as a result.

Case Study: Rotating Channel Flow

Model: RST with Elliptic relaxation

Reference: Wizman, V., D. Laurence, M. Kanniche, P. Durbin, and A. Demuren. "Modeling near-wall effects in second-moment closures by elliptic relaxation." International journal of heat and fluid flow 17, no. 3 (1996): 255-266.

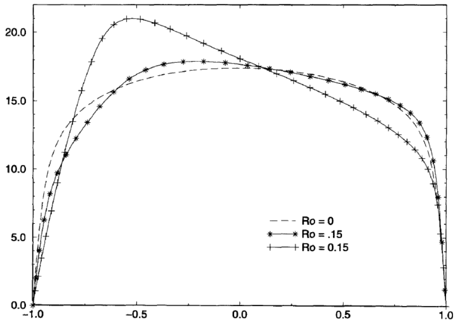


Figure 14 Mean velocity profiles for increasing rotation rates

Case Study: Rotating Channel Flow

Model: RST with Elliptic relaxation
Reference: Wizman, V., D. Laurence, M. Kanniche, P. Durbin, and A. Demuren. "Modeling near-wall effects in second-moment closures by elliptic relaxation." International journal of heat and fluid flow 17, no. 3 (1996): 255-266.

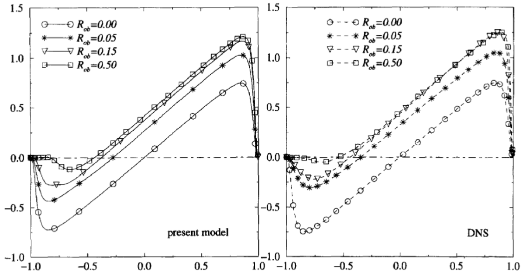


Figure 13 Shear stresses for increasing rotation rates

Case Study: Flow over Backward Facing Step

Model: $v^2 - f$

Reference: Durbin, P. A. "Separated flow computations with the k-epsilon-v-squared model." AIAA journal 33.4 (1995): 659-664.

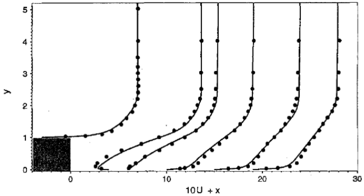


Fig. 3 Mean velocity profiles for the Jovic and Driver experiment.

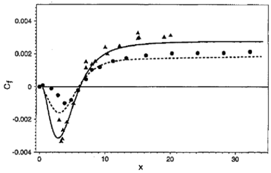


Fig. 2 Skin friction coefficient on wall downstream of the backstep compared to experiments of Jovic and Driver (——, ▲) and Driver and Seegmiller (– – –, ●).

Case Study: Flow in a Diffuser

Model: $v^2 - f$

Reference: Durbin, P. A. "Separated flow computations with the k-epsilon-v-squared model." AIAA journal 33.4 (1995): 659-664.

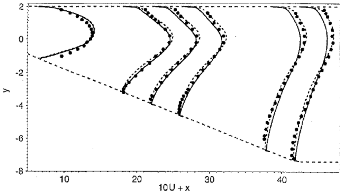


Fig. 8 Mean velocity profiles in the Obi et al. diffuser, both Eq. (9) (——) and a constant value of 1.55 (- - -) were used for $C_{\varepsilon_1}^{z}$, light dashed lines show the diffuser surface.

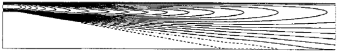


Fig. 9 Contours of U velocity component in the Obi et al. diffuser: (——) positive values; (- - -) negative values.

Case Study: Multiple Impinging Jets

Model: modified elliptic relaxation with RST model
Refernce: Thielen, L., K. Hanjali?, H. Jonker, and Remi Manceau. "Predictions of flow and heat transfer in multiple impinging jets with an elliptic-blending second-moment closure." International Journal of Heat and Mass Transfer 48, no. 8 (2005): 1583-1598.

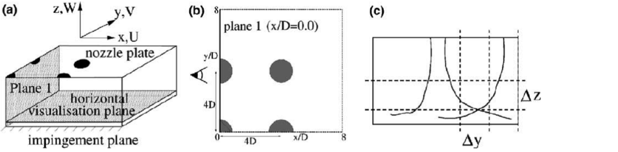


Fig. 1. The computational domain (one quarter of the flow) (a), the position of the planes on which the results are visualized (b) and a sketch of the location of the profiles presented (c). The eye indicates the view direction used to visualize the planes. Dashed lines indicate the lines along which the profiles are plotted.

Case Study: Multiple Impinging Jets

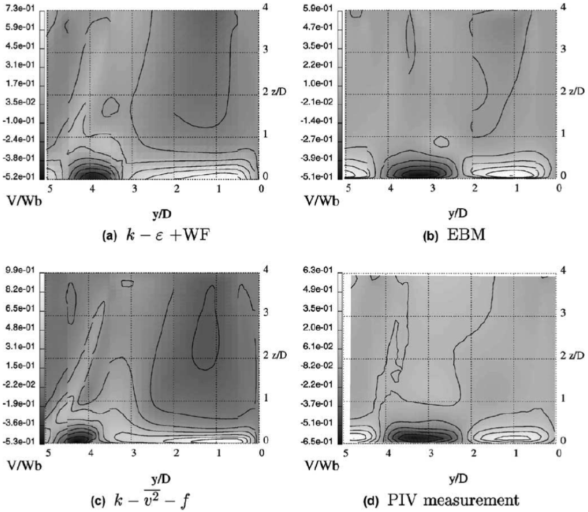


Fig. 5. Velocity component parallel to the impingement plane, scaled with the jet bulk velocity (V/W_b), in Plane 1 (x/D = 0.0). The wall jet of the undisturbed jet extends beyond y/D = 2.0.

Case Study: Multiple Impinging Jets

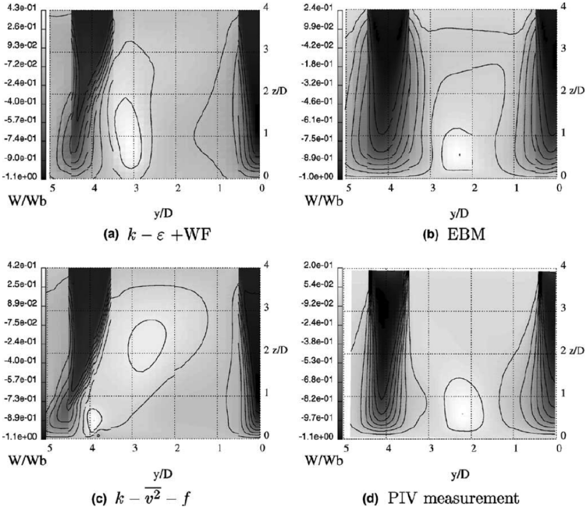
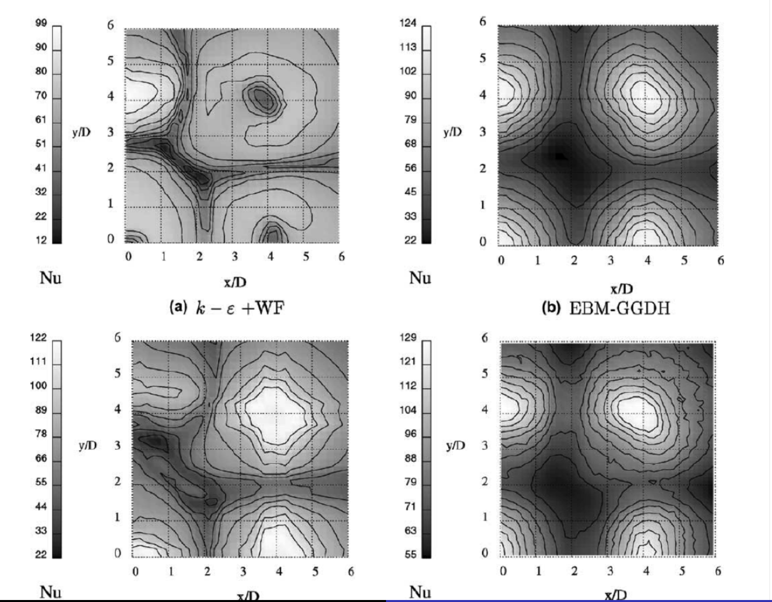


Fig. 4. Square set-up: contours of velocity component normal to the impingement plane, scaled with the jet bulk velocity (W/W_b), in Plane 1 ($x/D = 0.0$).

Case Study: Multiple Impinging Jets



Numerical Issues

Grid generation issues:

- For low-Re RANS models, make sure that grid point at wall has width ν/u_τ in *wall normal direction*.
- At the edge of boundary layer, wall normal grid size should be around $\delta/20$
- Wall-parallel grid size can always be around $\delta/20$
- Use square grid cells in free shear flows, with size $\delta/20$ (here δ is the width of the shear layer)
- Grid consistency tests can be conducted only for low-Re RANS models. For these tests, do not reduce the size of wall-paralel grid cells.
- For wall-modeled RANS, grid consistency tests cannot really be conducted, especially for the part of the grid near the wall.

Numerical Issues

Time stepping:

- Make sure you account for viscosity when calculating time step from CFL: $\Delta t = C_{CFL} \min[\Delta/U, \Delta^2/\nu]$. Near the wall, viscosity will determine time step.
- Better to use a fully implicit scheme for viscosity to get rid of this issue. Most solvers do not come with this feature.

Numerical discretization:

- Usual CFD discretization (e.g. finite difference, finite volume) may be used
- Usually upwind schemes can be used for RANS models, since the model itself may introduce a much larger uncertainty.

Numerical Issues

Most non-conservative terms will either be production $(+)$ or dissipation $(-)$ terms.

$$\frac{dk}{dt} = \pm\frac{k}{T}$$

Implicit time stepping: production terms can become singular (or stiff) at larger time steps

$$k^{n+1} = \frac{k^n}{1 \pm \Delta t/T}$$

Explicit time stepping: dissipation terms can introduce negative factor for large time steps

$$k^{n+1} = k^n(1 \mp \Delta t/T)$$

Numerical Issues

Rule of thumb: make dissipative terms implicit, production terms explicit. For example,

$$\frac{dk}{dt} = P - \epsilon$$

can be discretized as:

$$k^{n+1} = k^n + \Delta t \left(P^n - \frac{\epsilon^n}{k^n} k^{n+1} \right)$$

How do OpenFOAM/Ansys carry out the time stepping ? You should always check.

Numerical Issues

RST models are known to be numerically unstable. Reynolds stress components are coupled to each other.
For mean velocity equation, Reynolds stress appears as a body force (instead of an eddy diffusivity). One way to increase stability for U equation is to add and subtract diffusive terms:

$$\begin{aligned}\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_j}{\partial x_j} - \frac{\partial}{\partial x_j}[2\nu_T S_{ij}]^{(n+1)} &= -\frac{\partial P}{\partial x_i} + \nu \nabla^2 U_i \\ &\quad - \frac{\partial}{\partial x_j} R_{ij}^{(n)} - \frac{\partial}{\partial x_j}[2\nu_T S_{ij}]^n\end{aligned}$$

Numerical Issues

Boundary conditions for f, ϵ, ω can be stiff. They can be made less stiff by making the bc implicit. For example, at wall:

$$\epsilon^{(n)}|_{y=0} = \nu \frac{\partial^2 K}{\partial y^2} = 2\nu \lim_{y \to 0} \frac{K^{(n)}(y)}{y^2}$$

$y = y_1$ (y location of first off-wall grid point) is usually taken