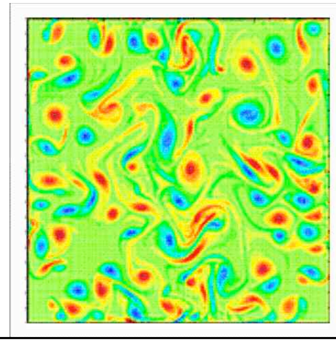
Vorticity and Turbulence
ME724: Essentials of Turbulence

(Based on "Turbulence and Vortex Dynamics" by Javier Jimenez)

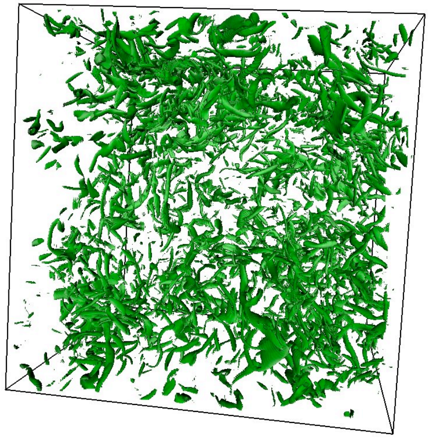
Instructor: Amitabh Bhattacharya

Introduction

- Turbulent flows contain large vorticity
  - Can be seen as vortex "tubes"
  - Varying size (circulation) and orientation
- What is the role of vorticity ?
  - Important for transferring energy between scales
  - Difference between 3D and 2D turbulence
    - Energy transferred from *smaller to larger scales* in 2D



2D Turbulence (vorticity)
TU Delft



Imagery produced by VAPOR (www.vapor.ucar.edu)

3D Turbulence
Kadoch, Polytechnic Univ, France

Decomposition of Velocity Field

- ❑Helmholtz Decomposition: We can decompose any vector field as follows:

$$\mathbf{u} = \nabla \times \Psi + \nabla \phi$$

- ❑ "Gauge" condition: $\nabla \bullet \Psi = 0$

$$\nabla \times \mathbf{u} = -\nabla^2 \Psi$$

$$\Psi = -\nabla^{-2} \omega$$

$$\mathbf{u} = -\nabla \times [\nabla^{-2} \omega] + \nabla \phi$$

$$\nabla \bullet \mathbf{u} = 0$$

$$\Rightarrow \nabla^2 \phi = 0$$

Vortical

Irrotational

2D Vortex Dynamics

- Biot-Savart Law in 2D

$$\boldsymbol{u} = \frac{1}{2\pi} \iint \omega(x', y') \frac{[y' - y,\, x - x']}{(x - x')^2 + (y - y')^2} \, \mathrm{d}x' \, \mathrm{d}y' + \nabla\phi$$

- Vorticity Equation in 2D $\quad \dfrac{\partial \omega}{\partial t} + \boldsymbol{u} \cdot \nabla\omega = \nu \nabla^2 \omega$

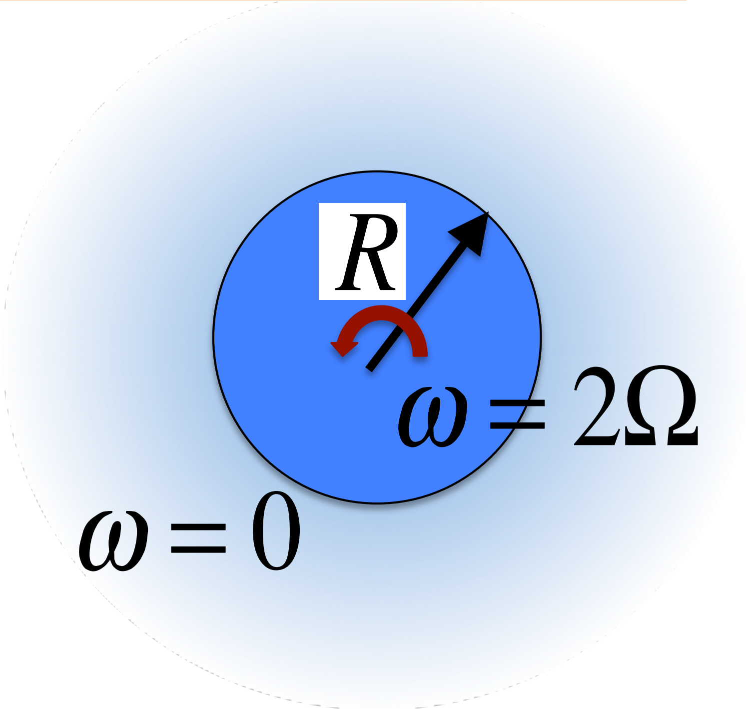
  - Vorticity is simply being advected by the flow, diffused by viscosity

- Flow can be complex:
  - Velocity advects vorticity
  - Vorticity affects velocity

Example: Rankine Vortex

- Rankine Vortex:
  - Solid body rotation in core
  - Irrotational flow outside core
  - Circulation outside (Kelvin's Theorem):

$$\Gamma = \oint \mathbf{u} \bullet \mathbf{ds} = 2\pi R^2 \Omega$$

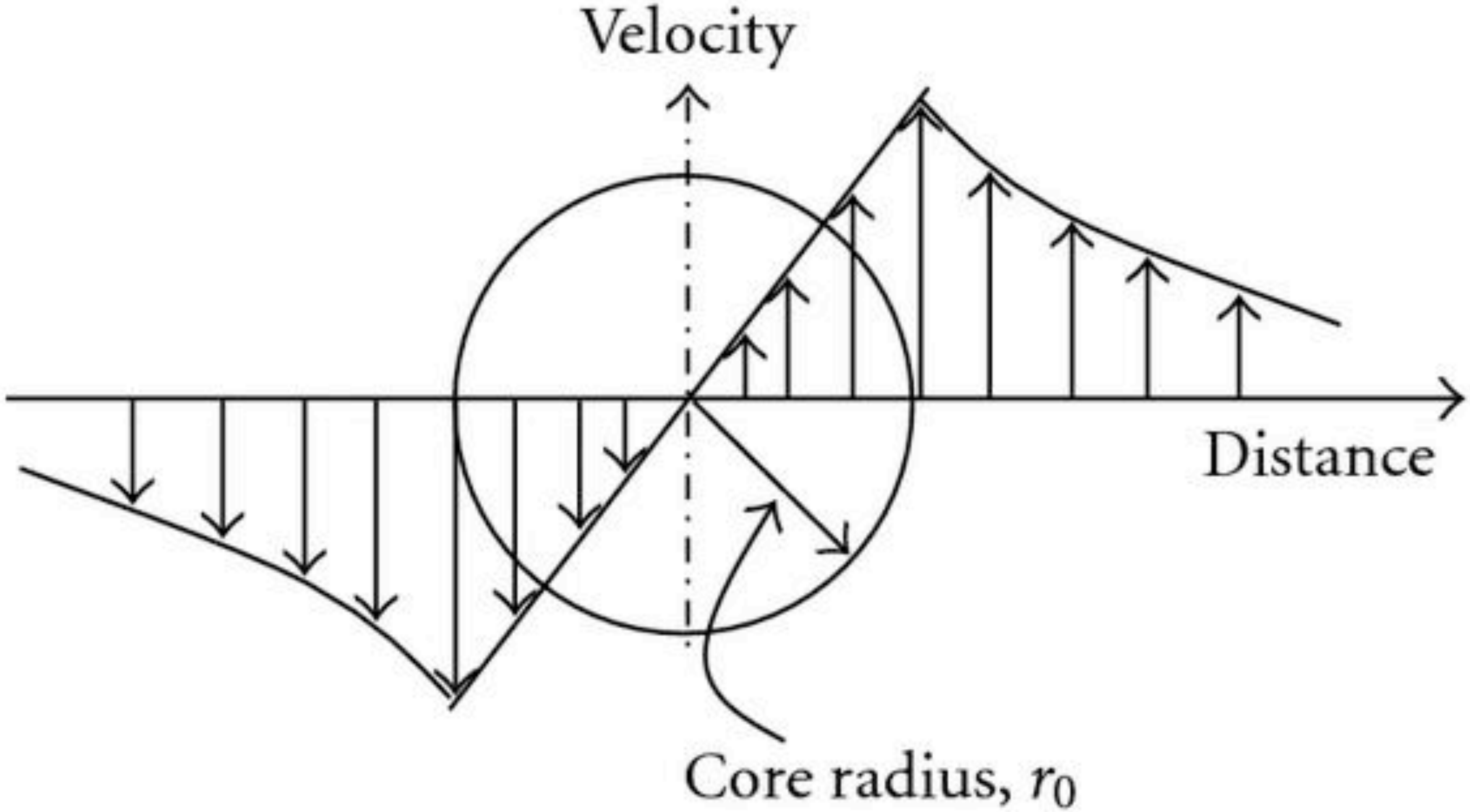


- Velocity field

$$u_r = 0 \qquad u_\theta = \begin{cases} \Gamma/(2\pi r) \text{ if } r > R \\ \Omega r \text{ if } r \le R \end{cases}$$

⇦ **Consistent with Biot-Savart law**

Rankine Vortex

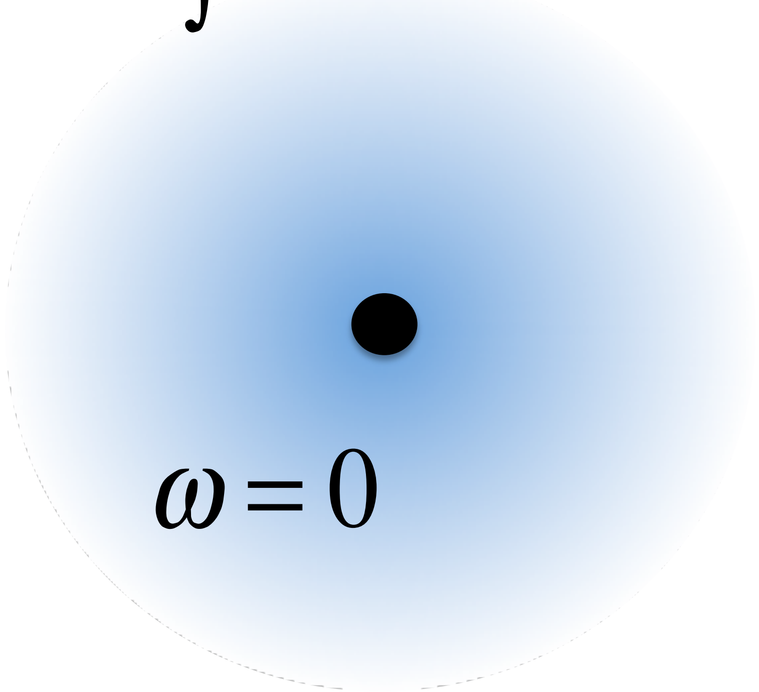


Point Vortices

- In the limit of $R \rightarrow 0$, we get a point vortex
- Good idealization far from the vortex
- At any point distance $\mathbf{r}$ from center, velocity is:

$$\mathbf{u} = \Gamma \frac{\mathbf{k} \times \hat{\mathbf{r}}}{2\pi r}$$

$$\Gamma = \oint \mathbf{u} \bullet \mathbf{ds} = 2\pi R^2 \Omega$$



Interaction Between Point Vortices

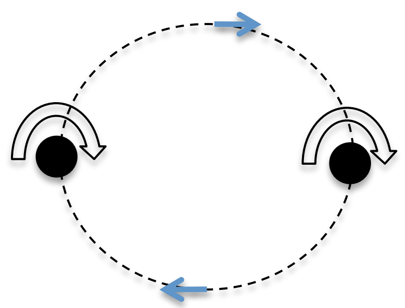
- For multiple point vortices:

$$\frac{d\mathbf{x}^i}{dt} = \mathbf{u}^i = \sum_{j \neq i} \Gamma^j \frac{\mathbf{k} \times [\mathbf{x}^i - \mathbf{x}^j]}{2\pi \left| \mathbf{x}^i - \mathbf{x}^j \right|^2}$$

  - Vortices are advected by local velocity field
  - Superposition possible, since velocity field is irrotational
  - Correct for infinite domain; different expression needed for finite/periodic domains

Interaction Between Point Vortices

- Vortices of same sign: Circle around each other



- Vortices of opposite sign: Travel in straight line

Chaos From Point Vortices

- Chaotic trajectory from more than 3 point vortices

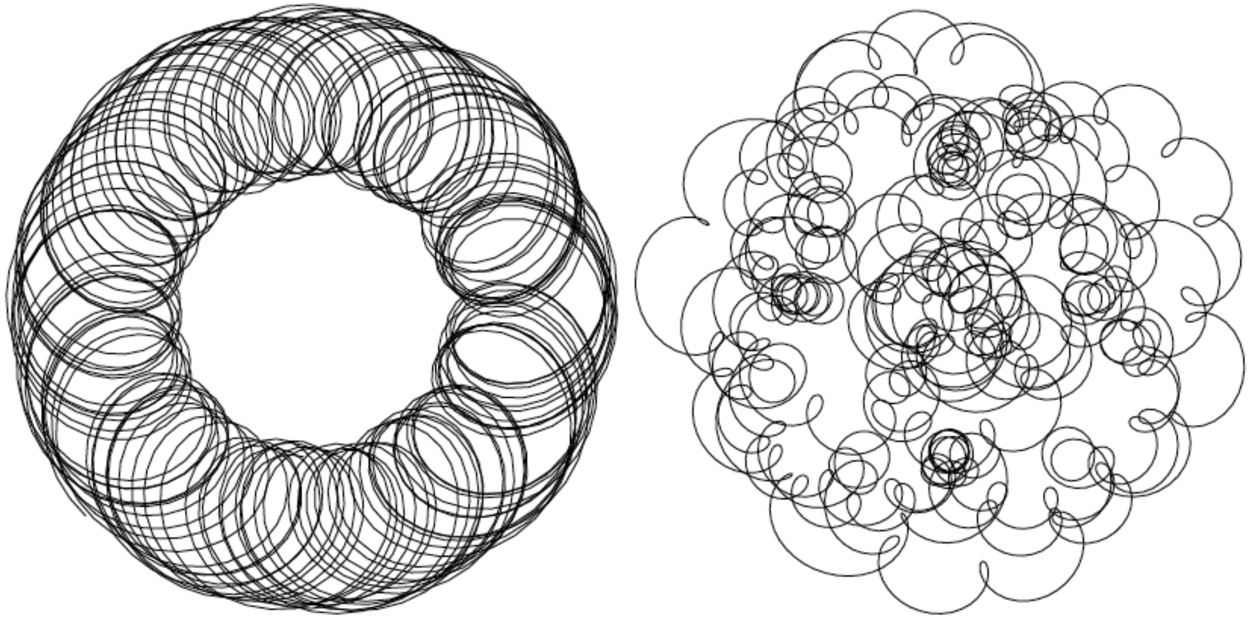


Figure 2.4: Trajectories of a vortex in a system of identical vortices initialized from random initial positions. Left: From a non-chaotic system of three vortices. Right: From a chaotic system of six vortices.

Chaos From Point Vortices

- Difference in evolution (Lyapunov exponent)

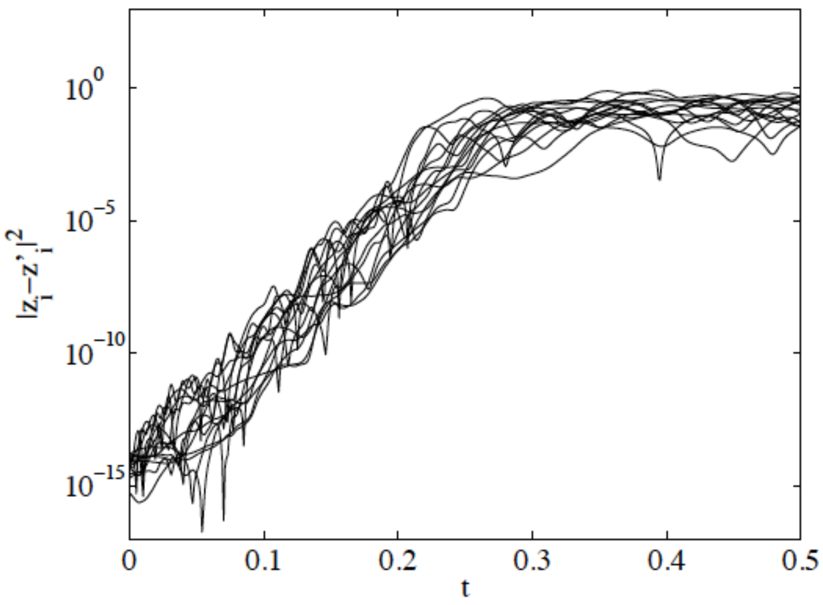


Figure 2.6: Evolution of the distance by which the corresponding vortices in two systems of fifteen vortices, whose initial conditions differ by random small amounts, diverge from one another. Each line corresponds to a different vortex pair. The initial exponential divergence is typical of chaotic systems, while the later saturation happens when the separations are large enough that the two systems can be considered uncorrelated from one another.

Finite-Sized 2D Vortices

- ❑Circular shape corresponds to a stable equilibrium
- ❑Self-induced rotation tends to make an elliptical vortex patch circular

Figure 2.1: Evolution of an initially elliptic Gaussian vortex, as it circularizes under its own induction. Time is from left to right. $\gamma/\nu = 1.2 \times 10^4$.

Interaction of Finite 2D Vortices

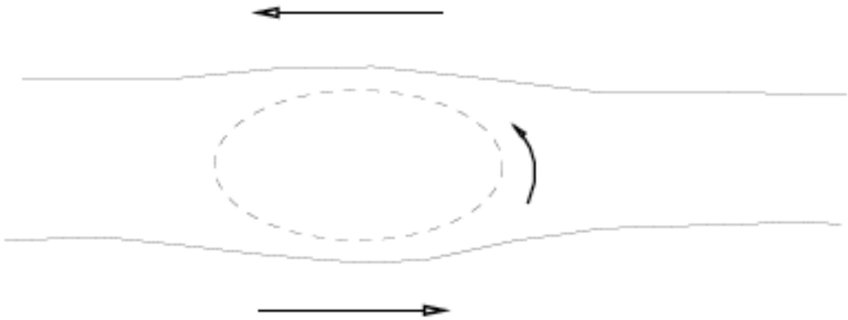
- Say we have a small and large vortex interacting (Circulation Γ_S and Γ_L, separated by distance r)
- When will large vortex deform small vortex ?
- Strain rate at distance *r*: $S \approx \Gamma_L/(2\pi r^2)$
- Angular velocity of small vortex: $\Omega \approx \Gamma_S/(\pi R^2)$
- For large vortex to affect small vortex

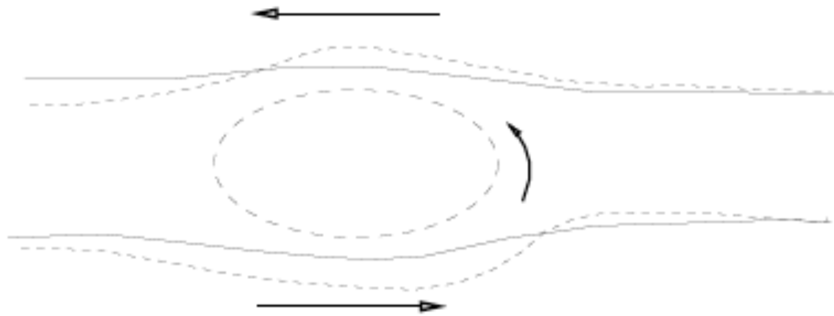
$$\Omega < S$$

$$\Rightarrow r < R\sqrt{\frac{\Gamma_L}{2\Gamma_S}}$$

Roll-Up Of Vortex Sheets

- Vortex sheet: Mixing layer with non-zero interface thickness
  - Several vortices placed in a line
  - Tend to roll up into large vortices (KHI)



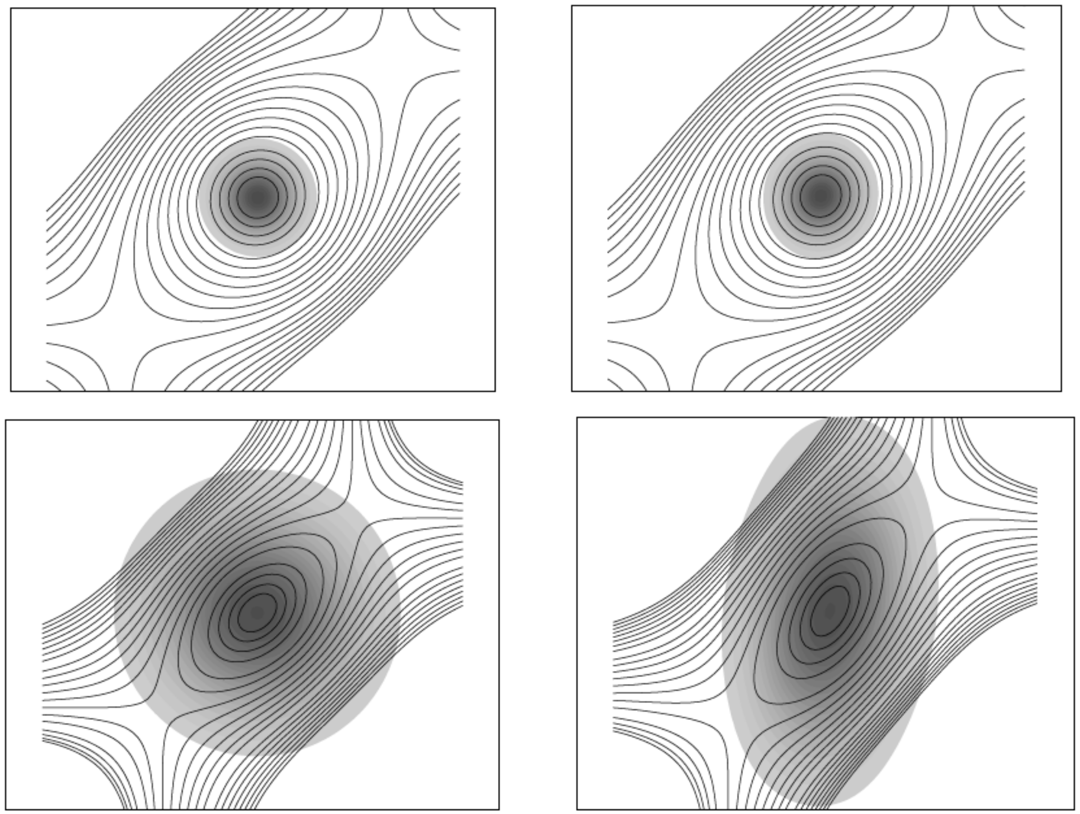


Vortex Mergers

- Large vortices tend to strain and absorb smaller vortices

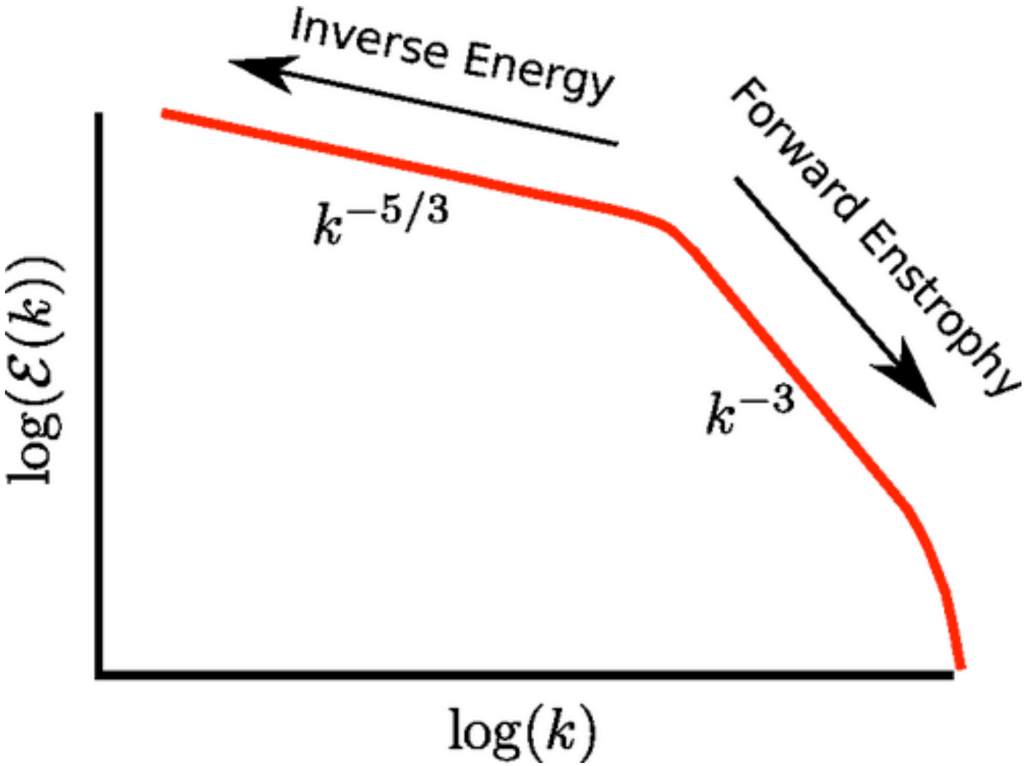
Vortex in Small Strain

Vortex in Large Strain



Inverse Cascade in 2D

- ❑Vortex mergers lead to *inverse* energy cascade

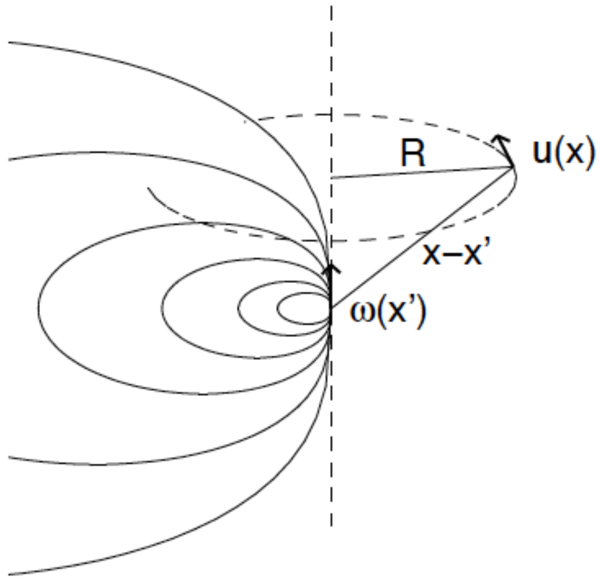


3D Vortices

- Biot Savart Law:

$$u(x) = \frac{1}{4\pi} \int \frac{\omega(x') \wedge (x - x')}{|x - x'|^3} \, \mathrm{d}^3 x' + \nabla\phi = u_\omega + \nabla\phi.$$

$$\nabla^2 \phi = 0.$$



- Vorticity Eqn:

Vortex Stretching Term

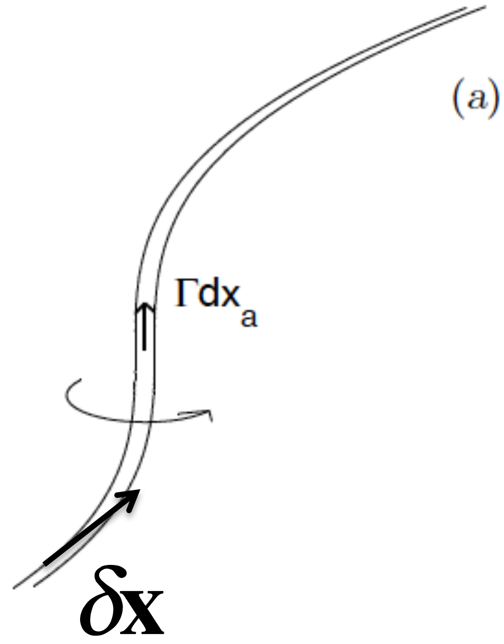
$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \underbrace{\mathbf{S} \cdot \omega}_{\text{Vortex Stretching Term}} + \nu \nabla^2 \omega$$

$$S_{ij} = \frac{1}{2}\left[u_{i,j} + u_{j,i}\right]$$

Vortex Tubes

- ❑ Vortex tubes
  - ➢ Surface of tube is parallel to vorticity vector
- ❑ Helmholtz's theorems:
  - ➢ Circulation (Γ) constant along tube
  - ➢ Tube cannot begin/end in fluid (inviscid)

    (From $\nabla \bullet \omega = 0$)
  - ➢ Tube is deformed along with material elements (inviscid)

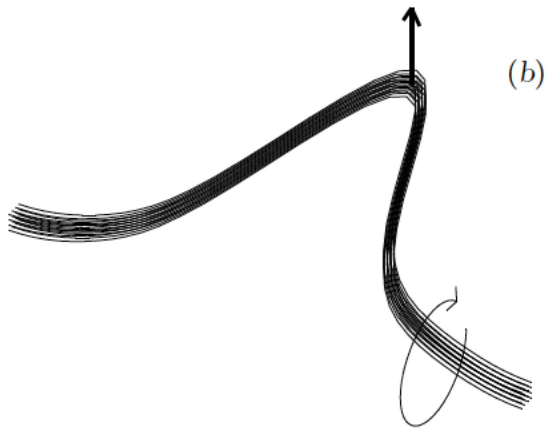


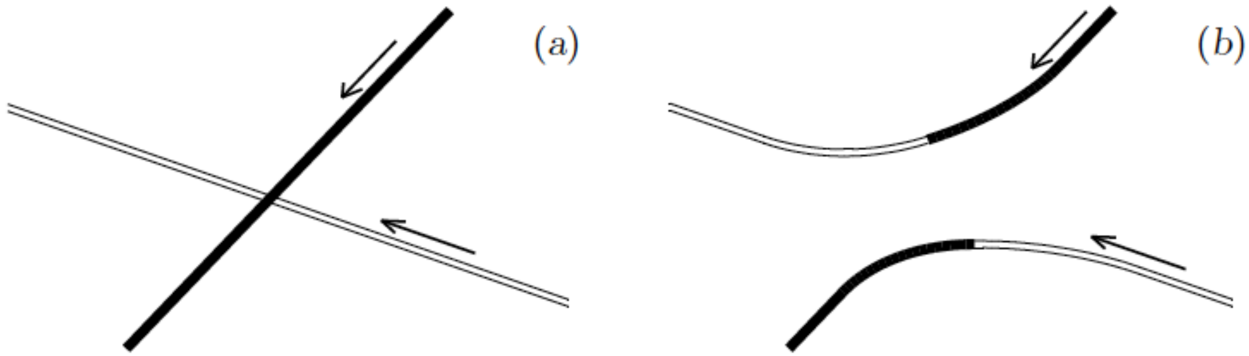
$$\frac{\partial \delta \mathbf{x}}{\partial t} + \mathbf{u} \cdot \nabla \delta \mathbf{x} = \mathbf{S} \cdot \delta \mathbf{x}$$

(same form as inviscid vorticity equation)

Vortex Interactions in 3D

- Self-induced rotation can cause out-of plane rotation of tube
- Interaction of two tubes
  - Two tubes can reconnect





Dissipation Rate Related to Vorticity

- Let's start with net viscous dissipation in volume

$$\varepsilon = \frac{\nu}{V}\int_V |\nabla \boldsymbol{u}|^2 \, \mathrm{d}^3\boldsymbol{x}.$$

- Use this identity:

$$|\nabla \boldsymbol{u}|^2 = |\boldsymbol{\omega}|^2 + (\nabla \cdot \boldsymbol{u})^2 + \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \nabla \cdot \boldsymbol{u}).$$

- If fluxes are zero at infinity (or box is periodic):

$$\varepsilon = \frac{\nu}{V}\int_V |\boldsymbol{\omega}|^2 \, \mathrm{d}^3\boldsymbol{x}. \qquad W = |\boldsymbol{\omega}|^2 = \omega_i \omega_i \quad \text{(Enstrophy)}$$

Enstrophy needs to be large for finite dissipation rate to exist

Enstrophy Eqn: 2D vs 3D

- Evolution of enstrophy (from vorticity eqn)

$$\mathrm{D}_t W = 2\boldsymbol{\omega} \cdot \boldsymbol{S} \cdot \boldsymbol{\omega} + \nu(\nabla^2 W - 2|\nabla \boldsymbol{\omega}|^2).$$

- In 3D, vortex stretching term present
  - Can increase/decrease enstrophy
  - Allows energy exchange between vortex and external straining field
- In 2D, stretching term is not present

Strain and Vorticity

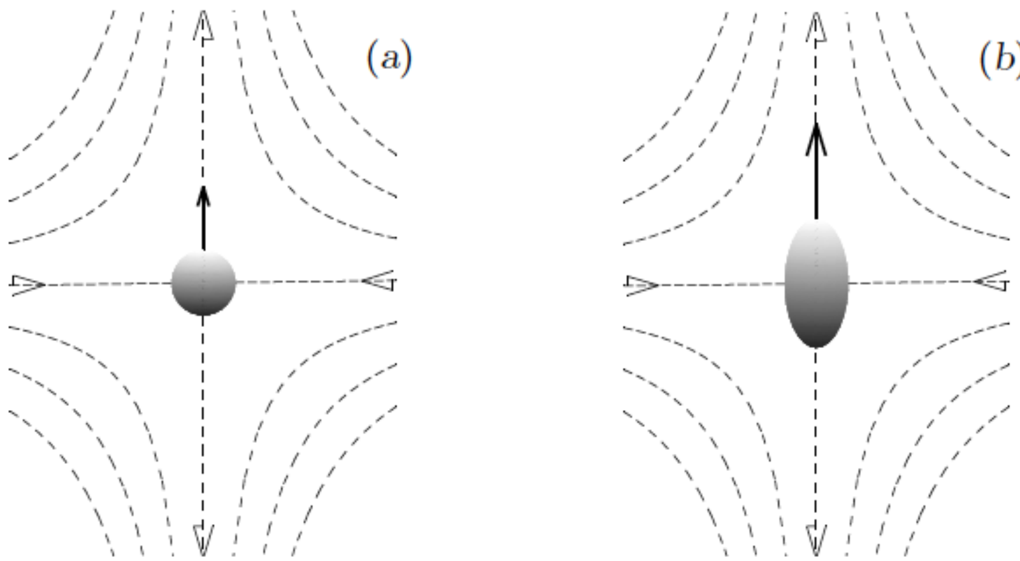
- Strain tensor has 3 eigenvalues and 3 eigenvectors: $\{\alpha_i, \mathbf{e}_i; i = 1,2,3\}$
  - Eigenvalues are real
  - Eigenvectors are orthogonal
  - Incompressibility implies: $\sum_i \alpha_i = 0$
- Choosing eigenvectors as coordinate axes:

$$D_t \omega_\beta = (\mathbf{S} \bullet \omega)_\beta = \alpha_\beta \omega_\beta$$

$$\omega_\beta(t) = \omega_\beta(0) \exp[\alpha_\beta t]$$

Why Strain Changes Vorticity

- ❑Stretching of material element changes its moment of inertia
- ❑Conservation of angular momentum leads to corresponding change in vorticity



Burger’s Vortex

- For very thin vortices, viscosity and strain rate can balance each other
- Consider imposed irrotational axisymmetric flow and axisymmetric vorticity:

$$U(\mathbf{x}) = \begin{bmatrix} \gamma x_1 \\ -\dfrac{\gamma}{2} x_2 \\ -\dfrac{\gamma}{2} x_3 \end{bmatrix}, \quad \gamma > 0, \quad \omega = \begin{bmatrix} \omega_1 \\ 0 \\ 0 \end{bmatrix}$$

Burger's Vortex

- Substituting into vorticity evolution equation:

$$U_2 \frac{\partial \omega_1}{\partial x_2} + U_3 \frac{\partial \omega_1}{\partial x_3} - \omega_1 \frac{\partial U_1}{\partial x_1} = \frac{1}{\mathrm{Re}} \left(\frac{\partial^2 \omega_1}{\partial x_2^2} + \frac{\partial^2 \omega_1}{\partial x_3^2} \right)$$

- Switching to polar coordinates:

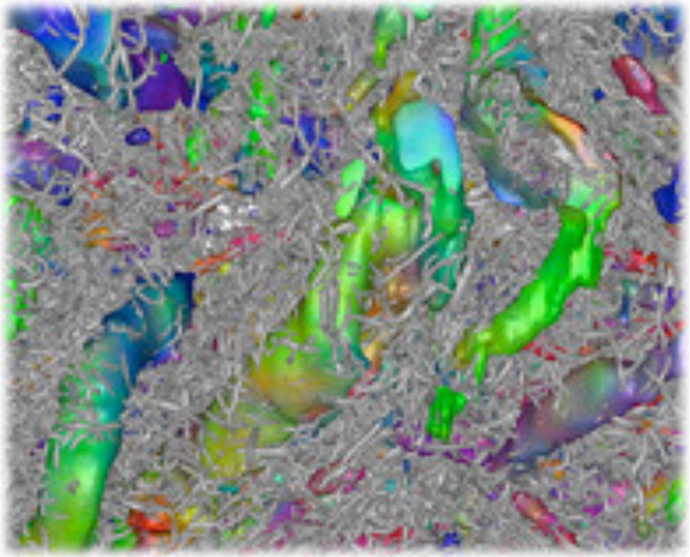
$$\frac{1}{\mathrm{Re}} \left(\frac{\partial^2 \omega_1}{\partial r^2} + \frac{1}{r} \frac{\partial \omega_1}{\partial r} \right) = -\gamma \left(\omega_1 + \frac{r}{2} \frac{\partial \omega_1}{\partial r} \right)$$

$$\text{Final Solution: } \omega_1(r) = A \exp[-\frac{\gamma \mathrm{Re}\, r^2}{4}]$$

Role of Vorticity in Energy Cascade

- Turbulence consists of range of scales
- Each scale contains vortex tubes
- On an average, vortex tube from larger scale strains vortices in smaller scales (Tennekes and Lumley)
  - Leading to “stretched” vortices, which break into even smaller vortices

Role of Vorticity in Energy Cascade



Burger *et. al*, Gallery of Fluid Motion, 65th APS-DFD Meeting