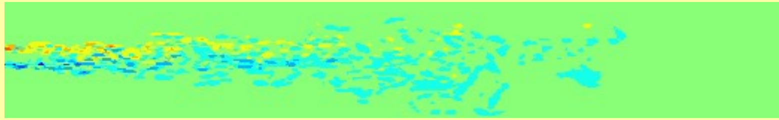
ME724: Two-equation Models



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Introduction

- Reynolds stress is related to mean velocity gradient via eddy viscosity
- Eddy viscosity requires time and length scale
- Mixing length model is a bit *adhoc*
  - Mean velocity gradient, mixing length used
  - Mixing length differs from case to case
- Turbulent Kinetic Energy (K) and Dissipation Rate (ε) can be used to get time and length scales instead
- Two-eqn modeling: Solve equations for $K(\mathbf{x},t)$ and $\varepsilon(\mathbf{x},t)$ along with $U(\mathbf{x},t)$
  - $\nu_T \sim K^2/\varepsilon$

How do we derive equations for K and ε ?

Reynolds Stress Transport Equations

- Start with Reynolds decomposition

$$\mathbf{u} = \mathbf{U} + \mathbf{u}', \quad p = P + p'$$

- Substitute into NS equation:

$$\frac{D}{Dt}(U_i + u_i') = -\frac{\partial}{\partial x_i}(P + p') + \nu \nabla^2 (U_i + u_i')$$

- Averaging gives mean eqn:

$$\frac{DU_i}{Dt} = -\frac{\partial P}{\partial x_i} + \nu \nabla^2 U_i - \frac{\partial R_{ij}}{\partial x_j}$$

- Subtracting exact and mean equations:

$$\frac{\partial u_i'}{\partial t} + U_k \frac{\partial u_i'}{\partial x_k} + u_k' \frac{\partial U_i}{\partial x_k} + \frac{\partial}{\partial x_k}(u_i' u_k' - R_{ik}) = -\frac{\partial P}{\partial x_i} + \nu \nabla^2 u_i'$$

Reynolds Stress Transport Equations

- We can write:

$$u_j'\left[\frac{\partial u_i'}{\partial t}+U_k\frac{\partial u_i'}{\partial x_k}+u_k'\frac{\partial U_i}{\partial x_k}+\frac{\partial}{\partial x_k}(u_i'u_k'-R_{ik})=-\frac{\partial P}{\partial x_i}+\nu\nabla^2 u_i'\right]$$

$$u_i'\left[\frac{\partial u_j'}{\partial t}+U_k\frac{\partial u_j'}{\partial x_k}+u_k'\frac{\partial U_j}{\partial x_k}+\frac{\partial}{\partial x_k}(u_j'u_k'-R_{jk})=-\frac{\partial P}{\partial x_j}+\nu\nabla^2 u_j'\right]$$

- Adding the above equations, averaging..

$$\overline{u_i'\frac{\partial u_j'}{\partial t}+u_j'\frac{\partial u_i'}{\partial t}}+\overline{U_k u_i'\frac{\partial u_j'}{\partial x_k}+U_k u_j'\frac{\partial u_i'}{\partial x_k}}+\ldots.=\ldots$$

$$\Rightarrow\frac{\partial\overline{u_i'u_j'}}{\partial t}+U_k\frac{\partial\overline{u_i'u_j'}}{\partial x_k}=\ldots$$

Reynolds Averaged Navier-Stokes Eqns

Reynolds Decomposition: $u_i = U_i + u'_i\,, \quad p = P + p'$

$$\boxed{\begin{aligned} U_i &= \overline{u}_i \\ P &= \overline{p} \end{aligned}}$$

Mean Momentum: $\boxed{\partial_t U_i + U_j \partial_j U_i = -\partial_i P + \nu \partial_j \partial_j U_i - \left(\overline{u'_i u'_j}\right)_j}$

Mean Continuity: $U_{i,i} = 0$

Reynolds Stress Transport Equations:

$$\partial_t \overline{u'_i u'_j} + U_k \partial_k \overline{u'_i u'_j} = \underbrace{-(\overline{u'_j \partial_i p'} + \overline{u'_i \partial_j p'})}_{(1)} \underbrace{- 2\nu \overline{\partial_k u'_i \partial_k u'_j}}_{(2)} \underbrace{- \partial_k \overline{u'_k u'_i u'_j}}_{(3)}$$
$$\underbrace{-\left[\overline{u'_j u'_k} \partial_k U_i + \overline{u'_i u'_k} \partial_k U_j\right]}_{(4)} \underbrace{+ \nu \nabla^2 \overline{u'_i u'_j}}_{(5)}$$

1. Pressure redistribution
2. Viscous dissipation
3. Turbulent transport
4. Production
5. Viscous transport

$$k = \frac{1}{2}\overline{u'_i u'_i}, \quad \Rightarrow \frac{Dk}{Dt} = \frac{D}{Dt}\left(\frac{1}{2}\overline{u'_i u'_i}\right)$$

Exact Equation for TKE

- Mean equation:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \nu \nabla^2 U_i - \frac{\partial R_{ij}}{\partial x_j}$$

- Kinetic energy:

$$\frac{\partial k}{\partial t} + U_j k_{,j} = -\left(\overline{p' u'_j}\right)_{,j} - \frac{1}{2}\left(\overline{u'_j u'_i u'_i}\right)_{,j} + \nu \nabla^2 k - \overline{u'_i u'_j} U_{i,j} - \nu \overline{u'_{i,j} u'_{i,j}}$$

$$= \Gamma_{k,k} + \nu \nabla^2 k + P - \varepsilon$$

Turbulent+Pressure transport: $\Gamma_i = -\overline{p' u'_i} - \frac{1}{2}\overline{u'_i u'_j u'_j}$

Production: $P = -\overline{u'_i u'_j} U_{i,j}$ Dissipation: $\varepsilon = \nu \overline{u'_{i,j} u'_{i,j}}$

Kinetic Energy Budget : Channel Flow

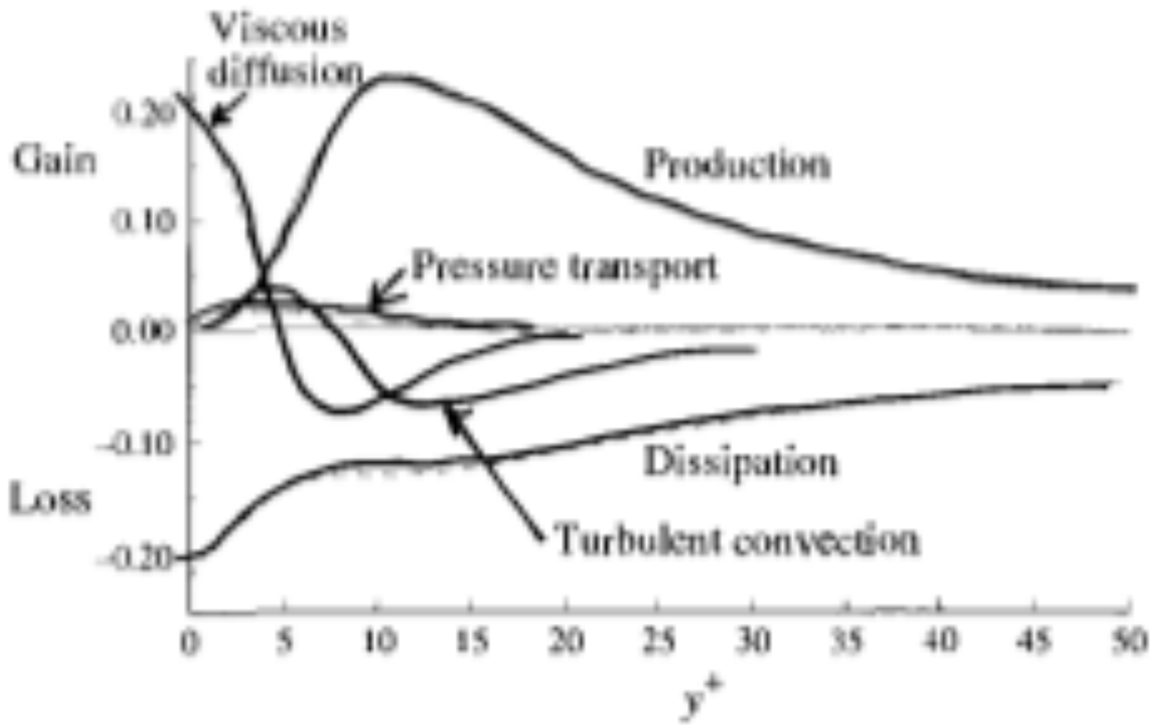


Fig. 7.18. The turbulent-kinetic-energy budget in the viscous wall region of channel flow: terms in Eq. (7.64) normalized by viscous scales. From the DNS data of Kim *et al.* (1987). Re = 13,750.

Kinetic Energy Budget: Round Jets

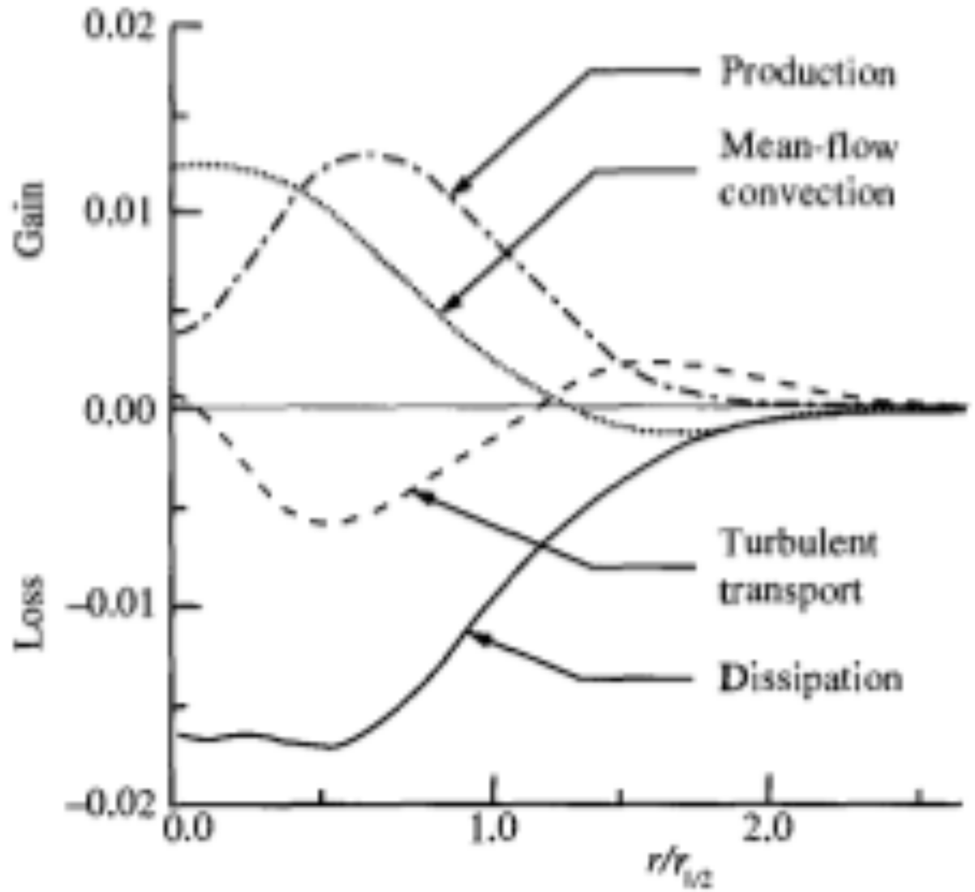


Fig. 5.16. The turbulent-kinetic-energy budget in the self-similar round jet. Quantities are normalized by U_0 and $r_{1/2}$. (From Panchapakesan and Lumley (1993a).)

Eddy Viscosity Models

❑Reynolds stress:

$$\overline{u_i u_j} = \tfrac{2}{3} k\delta_{ij} - 2\nu_T S_{ij}; \quad S_{ij} = \tfrac{1}{2}\left[U_{i,j} + U_{j,i}\right]$$

❑Mean momentum equation:

$$\boxed{\partial_t U_i + U_j \partial_j U_i = -\partial_i (P + \tfrac{2}{3} k) + \partial_j [\left(\nu + \nu_T\right)(\partial_j U_i + \partial_i U_j)]}$$

$$\nu_T = \nu_T (k, \varepsilon, \ldots)$$

Transport Term in k Eqn

- Transport term given by:

$$\Gamma_i = -\overline{p' u'_i} - \frac{1}{2}\overline{u'_i u'_j u'_j}$$

- Pressure term usually small
- Use eddy viscosity for scalar transport
- Second term on RHS:

$$\overline{u'_j u'_j u'_i} = \overline{\phi' u'_i}, \quad \phi' = u'_j u'_j$$

$$\overline{\phi' u'_i} = -C_\phi \nu_T \bar{\phi}_{,i} \Rightarrow \overline{u'_j u'_j u'_i} = -\nu_T \frac{\partial \overline{u'_j u'_j}}{\partial x_i}$$

$$\Gamma_i = \nu_T \frac{\partial k}{\partial x_i}$$

Production Term in k Eqn

- Production term given by:

$$P = -\overline{u'_i u'_j} U_{i,j}$$

- From "mixing length" hypothesis: Reynolds stress and shear rate usually of opposite sign
  - Production is usually positive
- From eddy viscosity model:

$$\overline{u_i u_j} = \tfrac{2}{3} k \delta_{ij} - 2\nu_T S_{ij}; \quad S_{ij} = \tfrac{1}{2}\left[U_{i,j} + U_{j,i}\right]$$

$$P = 2\nu_T S_{ij} S_{ij} > 0$$

The k-ε model

$$\boxed{\partial_t k + U_j \partial_j k = P - \varepsilon + \partial_j [(\nu + \nu_T)\partial_j k]}$$

$$\boxed{\partial_t \varepsilon + U_j \partial_j \varepsilon = \frac{C_{\varepsilon 1} P - C_{\varepsilon 2} \varepsilon}{T} + \partial_j \left[(\nu + \frac{\nu_T}{\sigma_\varepsilon}) \partial_j \varepsilon \right]}$$

$$\nu_T = C_\mu k^2 / \varepsilon, \qquad P = -\overline{u_i u_j} \partial_j U_i, \quad T = k / \varepsilon$$

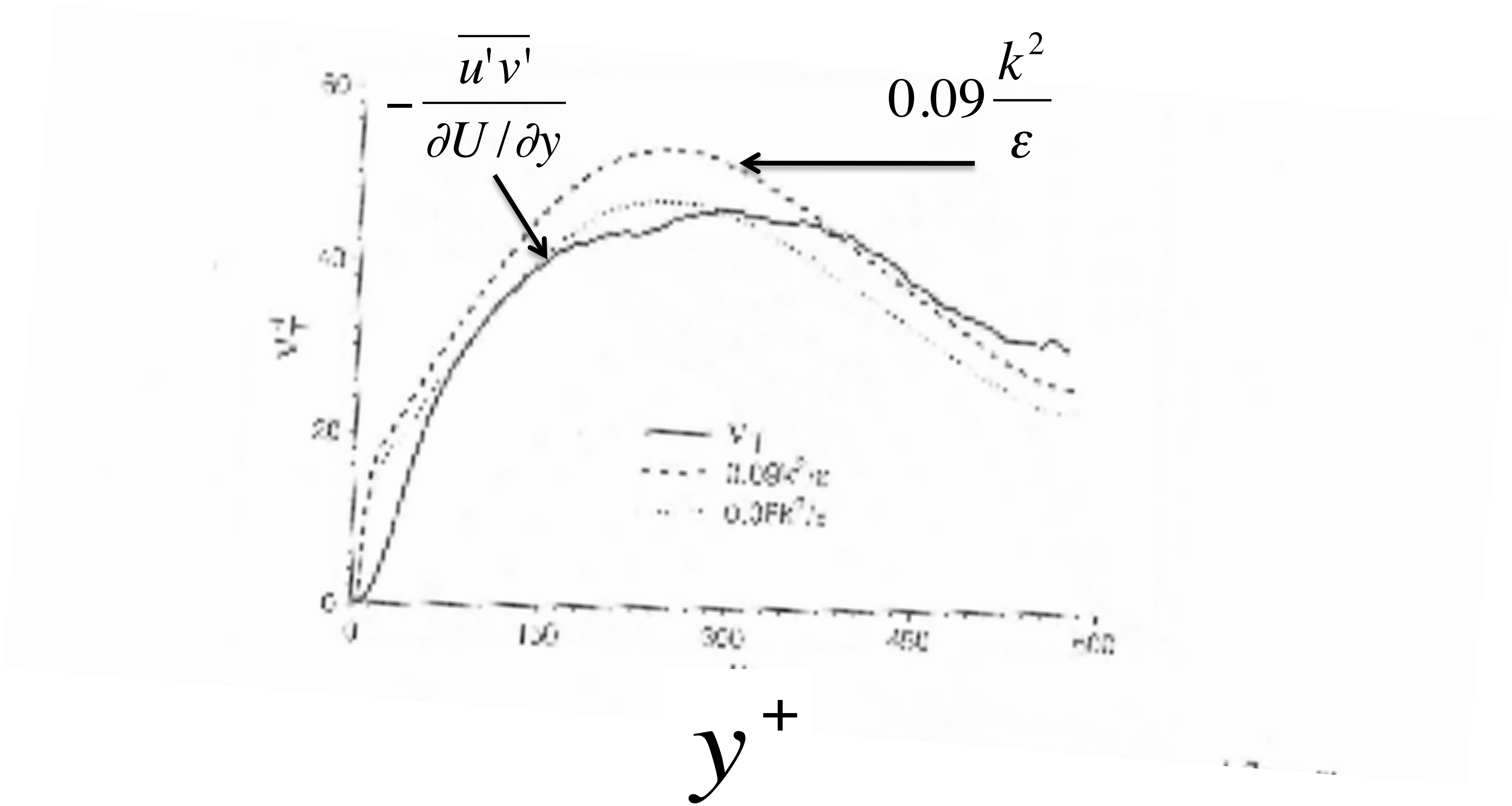
$$C_\mu = 0.09; \quad C_{\varepsilon 1} = 1.44; \quad C_{\varepsilon 2} = 1.92; \quad \sigma_\varepsilon = 1.3$$

BC at wall:

$$k = 0, \quad \text{or } \partial_n k = 0$$

$$\text{or} \quad \varepsilon = \nu \partial_n^2 k$$

ν_T: Model vs DNS



Damping of T, ν_T

- 1/T should not become singular at wall, therefore it is damped:

$$T = \max(k/\varepsilon, 6\sqrt{\nu/\varepsilon})$$

- Near the wall, k does not control the energy transfer, R_{22} does
  - $R_{22} \sim O(y^4)$ near wall, so damp eddy viscosity

$$\nu_T = f_\mu C_\mu \frac{k^2}{\varepsilon}, \quad f_\mu = \exp\left[\frac{-3.4}{(1 + R_T/50)^2}\right], \; R_T = \frac{k^2}{\nu\varepsilon}$$

The k-ω model

$$\partial_t k + U_j \partial_j k = 2\nu_T |S|^2 - k\omega + \partial_j [(\nu + \frac{\nu_T}{\sigma_k}) \partial_j k]$$

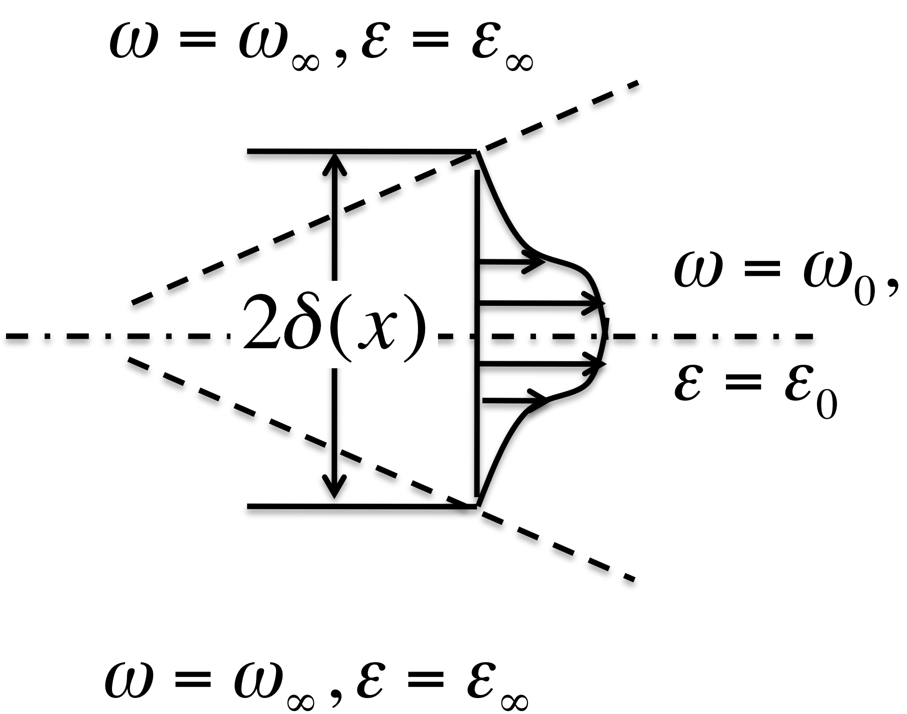
$$\partial_t \omega + U_j \partial_j \omega = 2C_\mu C_{\omega 1} |S|^2 - C_{\omega 2} \omega^2 + \partial_j [(\nu + \frac{\nu_T}{\sigma_\omega}) \partial_j \omega]$$

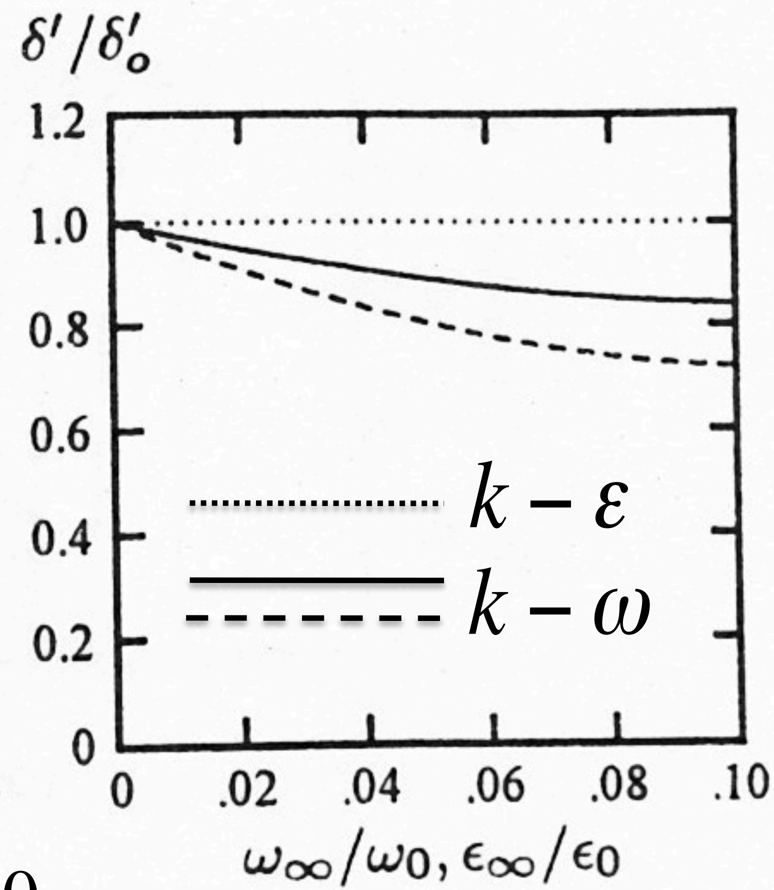
$$\omega = \frac{\varepsilon}{k} \qquad \nu_T = \frac{C_\mu k}{\omega}$$

$$S = \sqrt{S_{ij} S_{ij}}, \quad S_{ij} = (U_{i,j} + U_{j,i})/2$$

$$C_{\omega 1} = 5/9, \quad C_{\omega 2} = 5/6, \quad \sigma_\omega = \sigma_k = 2, \quad C_\mu = 0.09$$

Sensitivity to Free Shear Flow Spreading Rate





(c) Plane jet

$\delta' = \mathrm{d}\delta/\mathrm{d}x$

$\delta'_0 = \mathrm{d}\delta/\mathrm{d}x$ with $\omega_\infty = 0, \varepsilon_\infty = 0$

Far Wake Velocity Profile

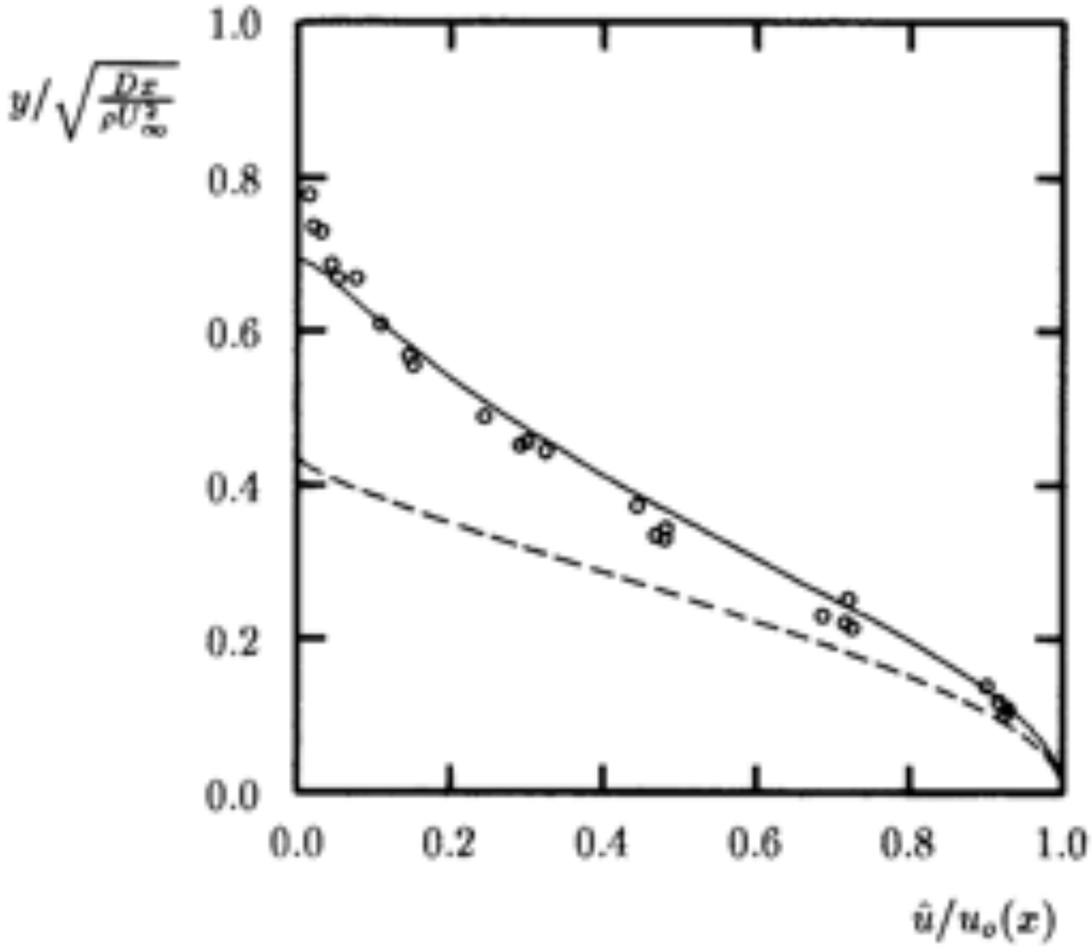


Figure 4.5: Comparison of computed and measured velocity profiles for the far wake; —— k-ω model; - - - k-ϵ model; o Fage and Falkner.

Spreading Rate in Free Shear Flow

Table 4.2: Free Shear Flow Spreading Rate

Flow	k-ω Model	k-ϵ Model	Measured
Far Wake	.301-.500	.256	.365
Mixing Layer	.103-.141	.098	.115
Plane Jet	.090-.136	.109	.100-.110
Round Jet	.073-.371	.120	.086-.095

2 Equation Models: Boundary Layers

Favorable Pressure Gradient

Adverse Pressure Gradient

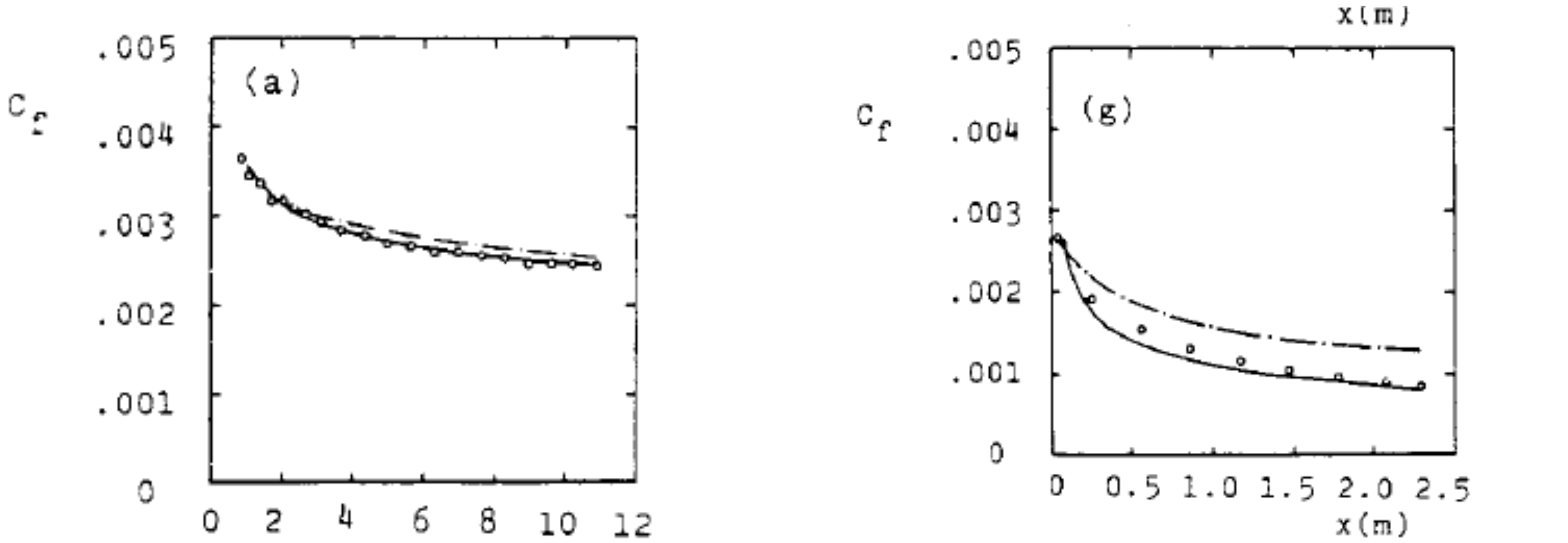


Figure 4.17: Computed and measured skin-friction and velocity profiles for incompressible boundary layers; —— k-ω model; – · – k-ϵ model; ∘ • measured. [From Wilcox (1988a) — Copyright © AIAA 1988 — Used with permission.]

Insensitivity to Rotation Rate

- In general, turbulence is either suppressed or amplified by rotation
- Both models are "frame-independent"

$$\mathbf{U}^{\text{Lab}}(\mathbf{x}) = \mathbf{U}^{\text{Rot}}(\mathbf{x}) + \boldsymbol{\Omega} \times \mathbf{x}$$

$$U_i^{\text{Lab}} = U_i^{\text{Rot}} + \varepsilon_{ijk}\Omega_j x_k \quad \Rightarrow S_{ij}^{\text{Lab}} = \left(U_{i,j}^{\text{Lab}} + U_{j,i}^{\text{Lab}}\right)/2 = S_{ij}^{\text{Rot}}$$

- But neither model is affected by rotation, since there is no effect of $\boldsymbol{\Omega}$

$$\overline{u_i u_j} = {}^2\!/_3\, k\delta_{ij} - 2\nu_T S_{ij}$$

Insensitivity to Rotation Rate

- In a rotating channel, mean velocity profile asymmetric
- Mean velocity equation:

$$0 = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial y}\left[\nu \frac{\partial U}{\partial y} - R_{12}\right]$$

$$0 = -\frac{\partial P}{\partial y} - \frac{\partial R_{22}}{\partial y} - 2\Omega U$$

- Rotation does not directly affect $U(y)$
  - Effect is on R_{12}

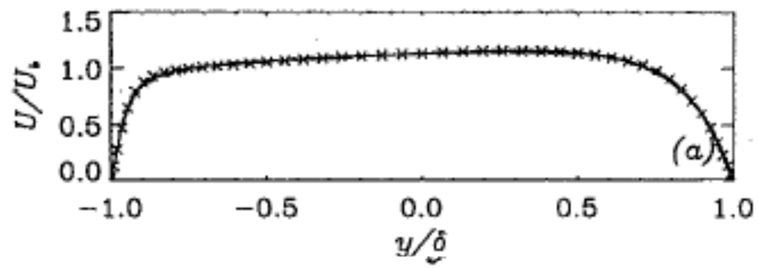


FIG. 1. Mean velocity in the rotating channel. $Re_b=5700$, $Ro_b=0.144$. —— First-order; - - - zeroth order; —·— plane-averaged; — — — 2.5 log $y^+ + 5.0$; × DNS (only every other point is shown). (a) Global coordinates; (b) wall coordinates, unstable side; (c) wall coordinates, stable side.

2-eqn models not capable of modeling effect of rotation

Stagnation Point Anomaly

- For 2 eqn models, TKE is severely overpredicted in stagnation flow
- Modeling of production is the issue $P = -R_{ij} S_{ij}$
- Pure strain rate implies

$$U = -Ax, \quad V = Ay$$

$$P = A(R_{11} - R_{22}) \quad \text{(Exact)}$$

$$P = 4\nu_T A^2 \text{ (Eddy Viscosity Model)}$$

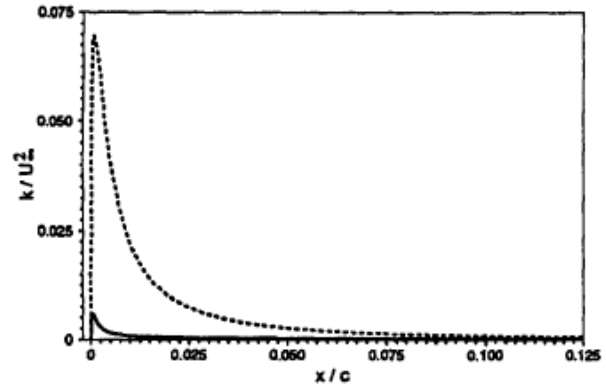


Figure 1 k/U_∞^2 along the stagnation streamline: ——, with 15 imposed; -----, without constraint

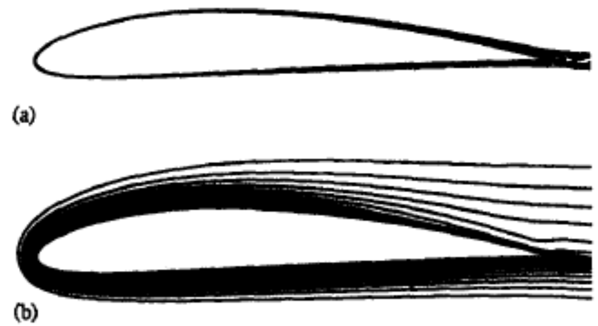


Figure 2 Contours of constant k/U_∞^2; (a), with 15 imposed; (b), without constraint; contour intervals of 1.5×10^{-3}

Model implies that production *always* increases quadratically with strain rate (not true for exact production); leads to overprediction of *k*

Stagnation Point Anomaly

- Durbin (1996) reduced anomaly by introducing "realizability" for eddy viscosity models
  - To ensure $0 < R_{\alpha\alpha} < 2k$, it must be true that:

$$2\nu_T \lambda^S_{\max} \leq \frac{2}{3}k$$

- Can also be shown that: $\lambda^S_{\max} < \sqrt{2|\mathbf{S}|^2/3}$
  - Implying $\nu_T \leq \dfrac{k}{\sqrt{6}|\mathbf{S}|}, \quad P \leq \dfrac{k|\mathbf{S}|}{\sqrt{6}}$
- Durbin therefore proposed the limiter

$$\nu_T = \min[C_\mu k^2/\varepsilon, k/(\sqrt{6}|\mathbf{S}|)]$$

Shortcomings of k-ε and k-ω

- Issues with k-ε
  - Requires damping function for eddy viscosity
  - Gives bad results (i.e. skin friction coefficient) for adverse pressure gradient
  - Implementation of ε BC at walls can be stiff
  - Wrong evolution rate for round jets
- Issues with k-ω model
  - For free shear flows, it is oversensitive to free-stream turbulence values
  - Gives $\varepsilon=0$ at the wall
  - Severely underpredicts kinetic energy in channel flows
- Issue with both models
  - Insensitive to rotation
  - Stagnation point anomaly

Menter's SST model

- Relation between k-ω and k-ε models

$$\varepsilon = k\omega \Rightarrow \frac{\partial \varepsilon}{\partial t} = k\frac{\partial \omega}{\partial t} + \omega\frac{\partial k}{\partial t}$$

$$\boxed{\partial_t \varepsilon + U_j \partial_j \varepsilon = \frac{C_{\varepsilon 1}P - C_{\varepsilon 2}\varepsilon}{T} + \partial_j\left[(\nu + \frac{\nu_T}{\sigma_\varepsilon})\partial_j \varepsilon\right] + S_\omega}$$

$$S_\omega = \frac{2}{T}\left(\nu + \frac{\nu_T}{\sigma_\omega}\right)\left[\frac{|\nabla k|^2}{k} - \frac{\nabla k \bullet \nabla \varepsilon}{\varepsilon}\right]$$

Menter's SST model

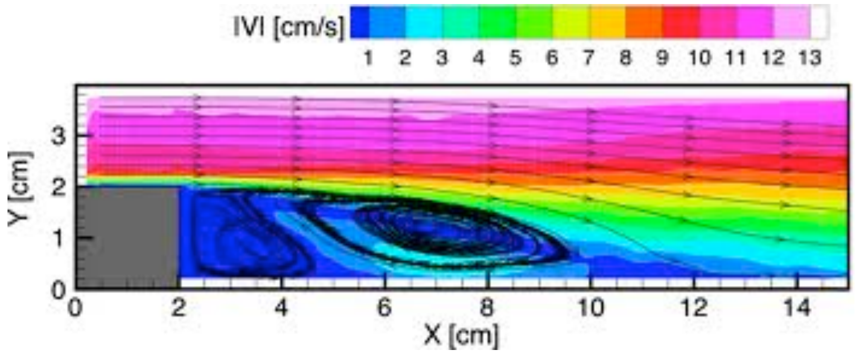
❑ Uses a "blending function" *F*

$$F = F(\nu, k, \omega, y, \nabla k, \nabla \omega)$$

- $F=0$ in the outer half of boundary layer
- $F=1$ in the inner half of boundary layer

$$\boxed{\partial_t \varepsilon + U_j \partial_j \varepsilon = \frac{C_{\varepsilon 1} P - C_{\varepsilon 2} \varepsilon}{T} + \partial_j \left[(\nu + \frac{\nu_T}{\sigma_\varepsilon}) \partial_j \varepsilon \right] + F S_\omega}$$

SST Model: Flow Over Backward Facing Step



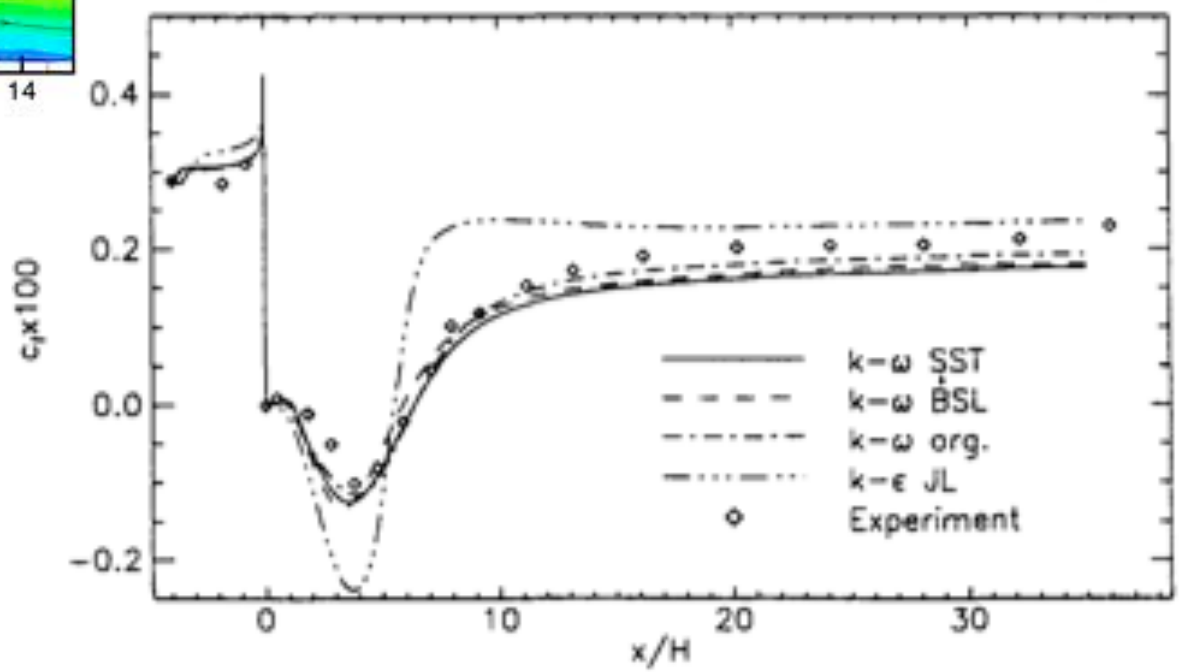


Fig. 17 Wall shear-stress distribution for backward-facing step flow.

The k-ε model: Where do model constants come from ?

$$\boxed{\partial_t k + U_j \partial_j k = P - \varepsilon + \partial_j [(\nu + \nu_T)\partial_j k]}$$

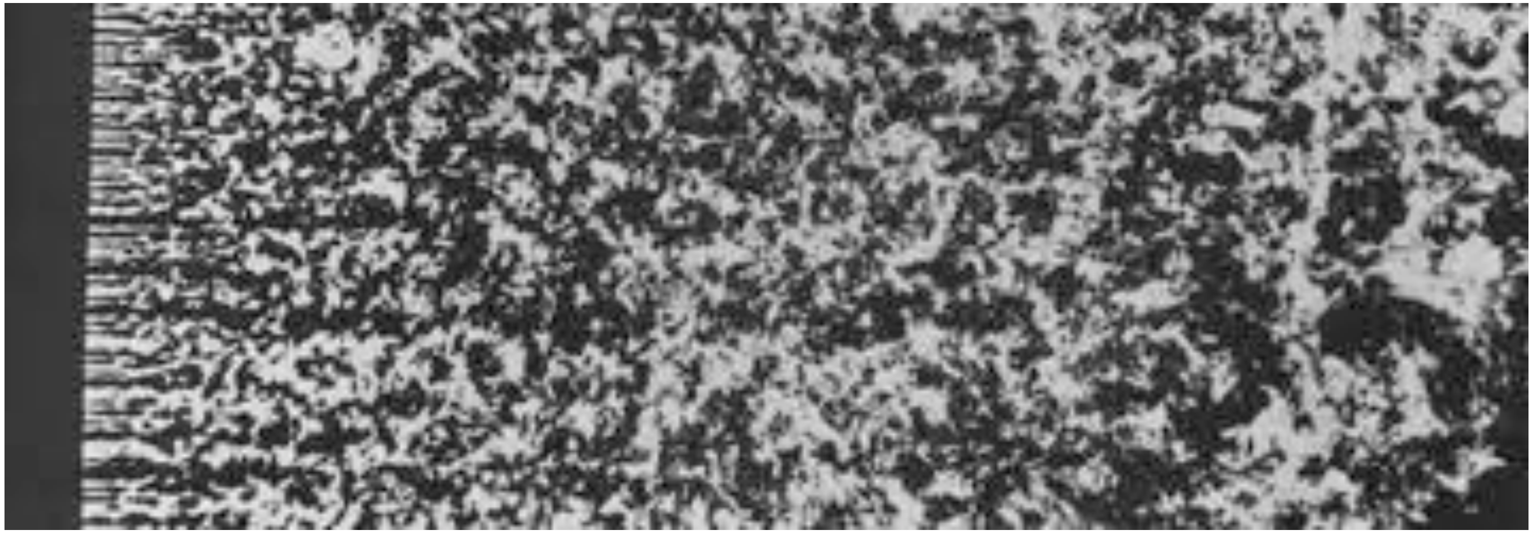
$$\boxed{\partial_t \varepsilon + U_j \partial_j \varepsilon = \frac{C_{\varepsilon 1} P - C_{\varepsilon 2}\varepsilon}{T} + \partial_j \left[(\nu + \frac{\nu_T}{\sigma_\varepsilon})\partial_j \varepsilon \right]}$$

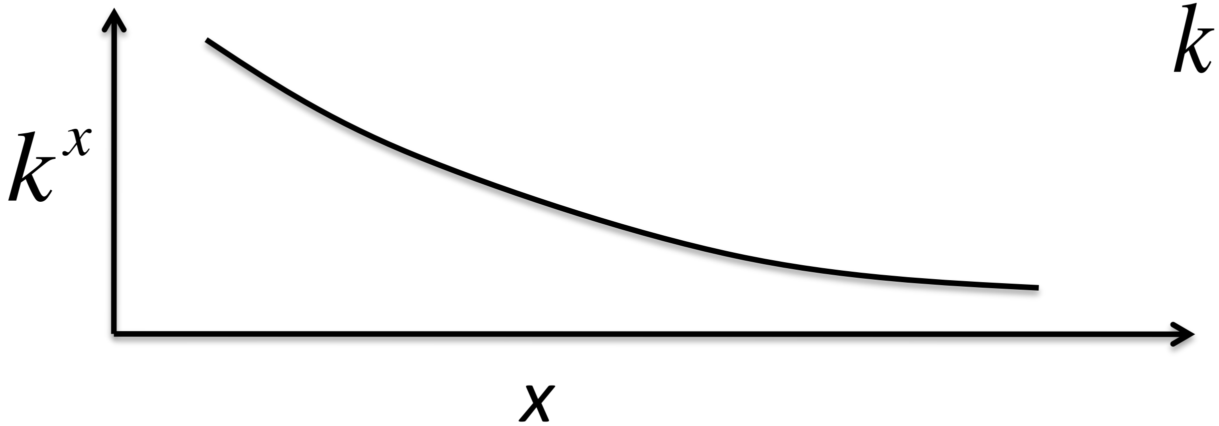
$$\nu_T = C_\mu k^2 / \varepsilon, \qquad P = -\overline{u_i u_j}\partial_j U_i, \quad T = k/\varepsilon$$

$$\boxed{C_\mu = 0.09}; \quad C_{\varepsilon 1} = 1.44; \quad \boxed{C_{\varepsilon 2} = 1.92;} \quad \sigma_\varepsilon = 1.3$$

Let's examine $C_{\varepsilon 2}$ and C_μ

k-ε Model Constants
$C_{\varepsilon 2}$: Decaying Grid Turbulence





$$k^x(x) = k^x(Ut)$$
$$= k^t(t)$$

k-ε Model Constants
$C_{\varepsilon 2}$: Decaying Grid Turbulence

- Isotropic turbulence with no mean shear:

$$\frac{dk}{dt} = -\varepsilon; \quad \frac{d\varepsilon}{dt} = -C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

- Seeking power law solution:

$$k = \frac{k_0}{(t/t_0+1)^n}; \quad \varepsilon = \frac{nk_0}{t_0(t/t_0+1)^{n+1}} \quad \text{(From } k \text{ eqn)}$$

- Substituting into ε eqn:

$$C_{\varepsilon 2} = \frac{n+1}{n}$$

- From experiments and simulations, $n=1.3$

$$\Rightarrow C_{\varepsilon 2} = 1.83$$

k-ε Model Constants
C_μ : The "Plateau" in k

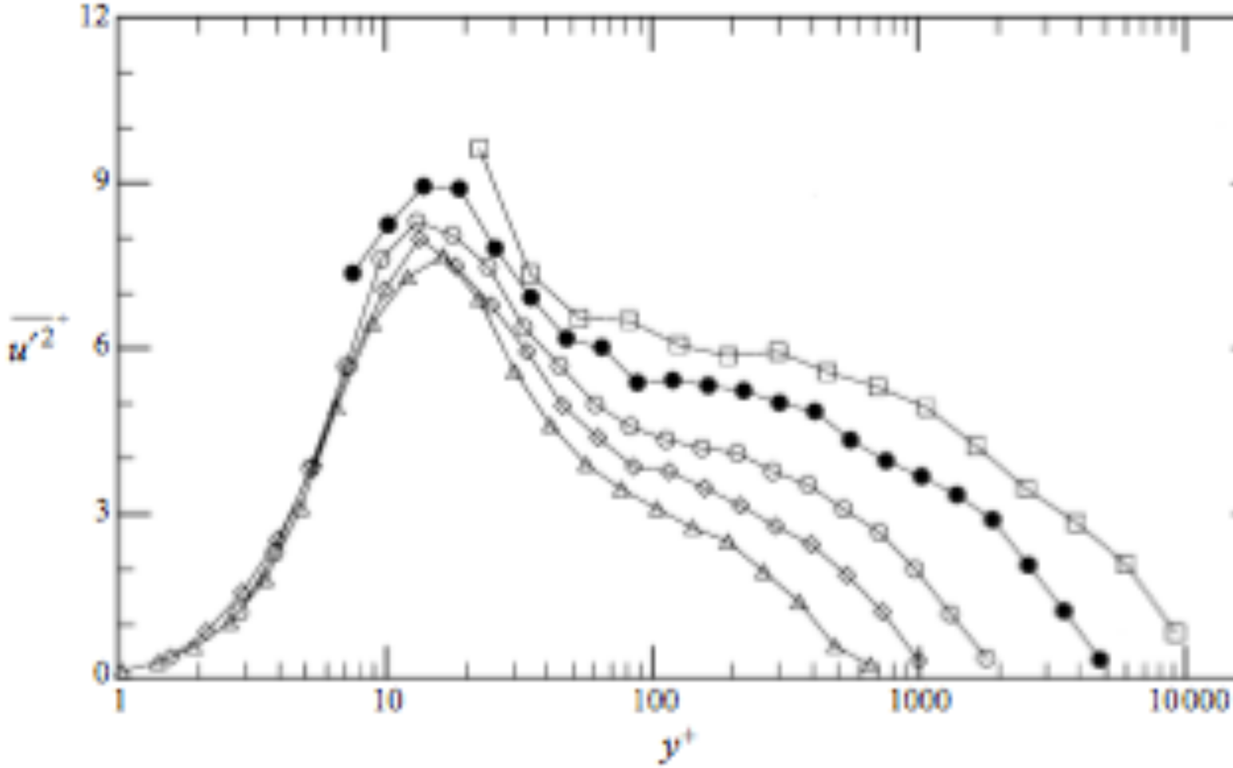


FIGURE 7. $\overline{u'^2}$ in inner–inner coordinates. Symbols in table 1.

$$\frac{\partial k}{\partial y} = 0 \Rightarrow P = \varepsilon \text{ in log region}$$

k-ε Model Constants
The value of C_μ

- Production=dissipation in log-region implies

$$\varepsilon = P = -R_{12}\frac{\partial U}{\partial y} = \frac{u_\tau^3}{\kappa y} \quad \text{(from log - law)}$$

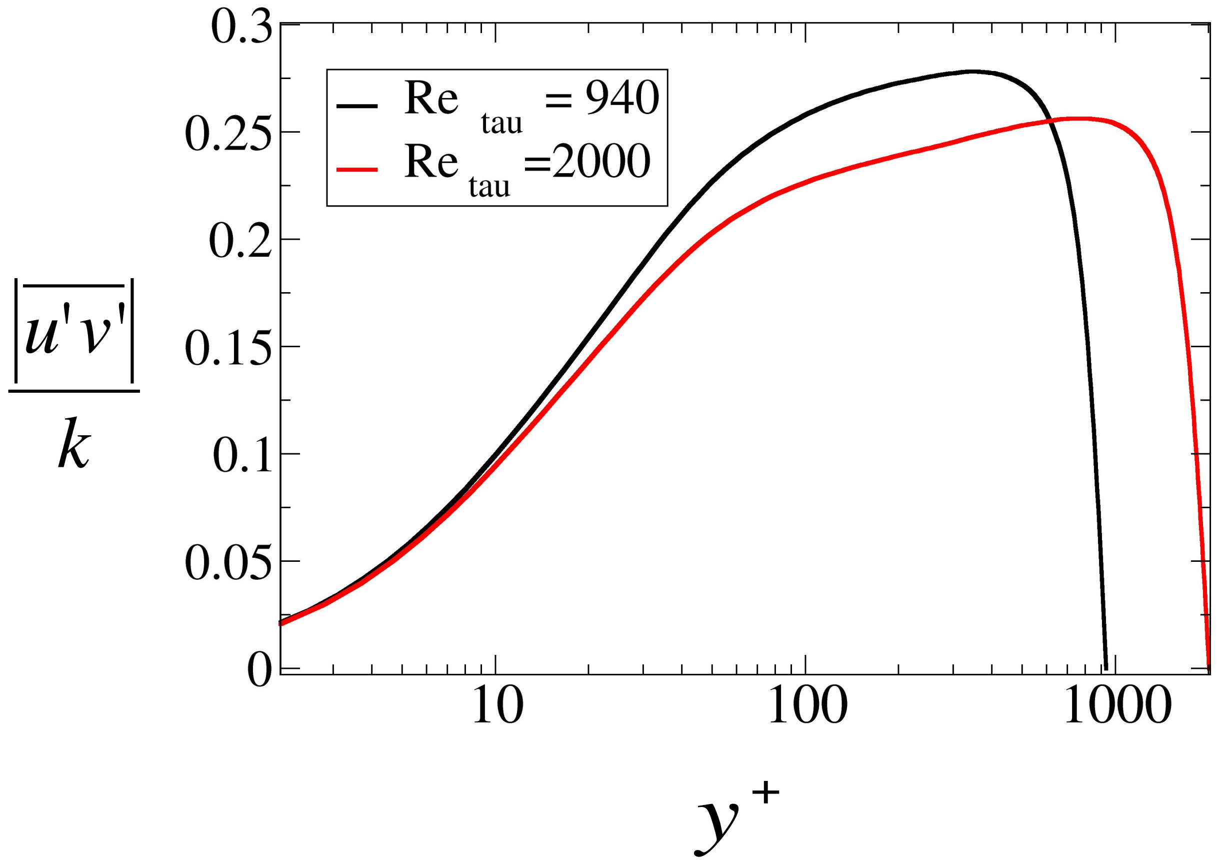
$$\nu_T = -\frac{R_{12}}{\partial U/\partial y} \quad \text{(definition of eddy viscosity)}$$

$$= \kappa y u_\tau \quad \text{(log - law)}$$

$$= \frac{C_\mu k^2}{\varepsilon} = \frac{C_\mu k^2 \kappa y}{u_\tau^3} \quad (k - \varepsilon \text{ model})$$

$$\Rightarrow k = \frac{u_\tau^2}{\sqrt{C_\mu}} \Rightarrow \quad C_\mu \sim \left(\frac{R_{12}}{k}\right)^2 \sim 0.09 \quad \left(\frac{R_{12}}{k} \sim 0.3 \quad \text{in log region}\right)$$

Ratio of Reynolds Stress w.r.t. KE



Wall Functions

- Provide boundary condition in log-region
  - Resolving viscous layer is expensive
  - Let y_p be the distance at which "patching" is done

$$y_p^+ > 50, \quad y_p / \delta < 0.2$$

- Based on "log-law", boundary conditions are :

$$\frac{dU}{dy} = \frac{u_\tau}{\kappa y_p}, k = \frac{u_\tau^2}{\sqrt{C_\mu}}, \varepsilon = \frac{u_\tau^3}{\kappa y_p}$$

- Find u_τ iteratively from "log-law":

$$U(y_p) = u_\tau [\frac{1}{\kappa} \log \frac{y_p u_\tau}{\nu} + B_i]$$

Sign of u_τ same as $U(y_p)$

Wall Functions

- But u_τ can change sign
  - Causes issues with BC for ε
- Instead, define new velocity scale u_k

$$u_k = \left(k\sqrt{C_\mu}\right)^{1/2}$$

- Modified BCs are:

$$\frac{dU}{dy} = \frac{u_\tau}{\kappa y_p}, \quad \frac{dk}{dy} = 0, \quad \varepsilon = \frac{u_\tau^3}{\kappa y_p}$$

- But ensuring that $y_p^+ > 50$ is a challenge