ME724: Essentials of Turbulence

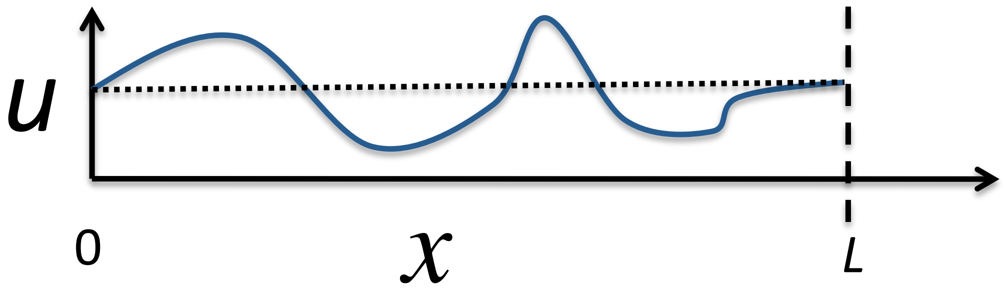
Lecture: Turbulent Time and Length Scales

Instructor: Amitabh Bhattacharya

Department of Mechanical Engineering
I.I.T. Bombay

Turbulence In a Box

- Let's consider a periodic box
  - *u(0)=u(L)*



- Possible to consider a Fourier expansion

$$u(x) = \sum_{\tilde{k}=-N/2}^{N/2} \hat{\tilde{u}}(\tilde{k}) \exp\left[\frac{2\pi i \tilde{k} x}{L}\right] = \sum_{k} \hat{u}(k) \exp[ikx]$$

$$\lambda = \frac{L}{\tilde{k}} = \text{Wavelength}, \quad k = \frac{2\pi \tilde{k}}{L} = \frac{2\pi}{\lambda} = \text{Wavenumber}$$

Properties of Fourier Series

- If *u(x)* is real then $\hat{u}(k) = \hat{u}^*(-k)$
- Distinct Fourier modes are orthogonal

$$\int_0^L \exp[ikx]\exp[ilx]\mathrm{d}x = L\delta_{k,-l}$$

- Correlations (**u**(**x**) statistically homogenous):

$$\overline{u} = \frac{1}{L}\int_0^L u\,\mathrm{d}x = \hat{u}(0)$$

$$\boxed{\overline{u^2}} = \frac{1}{L}\int_0^L u^2\mathrm{d}x = \sum_k \sum_{k'} \hat{u}(k)\hat{u}(k')\left[\frac{1}{L}\int_0^L \exp[ikx]\exp[ik'x]\mathrm{d}x\right]$$

$$= \sum_k \sum_{k'} \hat{u}(k)\hat{u}(k')\delta_{k,-k'} = \boxed{\sum_k \hat{u}(k)\hat{u}^*(k)}$$

Fourier Decomposition in 3D

- In 3D space, $\mathbf{k} = k_x\mathbf{i} + k_y\mathbf{j} + k_z\mathbf{k}$

$$u(\mathbf{x}) = \sum_{\mathbf{k}} \hat{u}(\mathbf{k}) \exp[i\mathbf{k}\cdot\mathbf{x}]$$

- Physical interpretation: $k = |\mathbf{k}| = \text{Wavenumber}$

$$\text{If } \ \mathbf{x} = s\frac{\mathbf{k}}{|\mathbf{k}|} = s\hat{\mathbf{k}} \quad \Rightarrow \quad \exp[i\mathbf{k}\cdot\mathbf{x}] = \exp[iks]$$

- Energy : $\overline{u^2} = \frac{1}{L^3}\int u^2 \mathrm{d}\mathbf{x} = \sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k})\cdot\hat{\mathbf{u}}^*(\mathbf{k}) = \sum_{i=1}^{M} E(k_i)\Delta k$

$$E(k_i) = \frac{1}{\Delta k} \sum_{\mathbf{k};(i-1)\Delta k \le k < i\Delta k} \hat{\mathbf{u}}(\mathbf{k})\cdot\hat{\mathbf{u}}^*(\mathbf{k})$$

2D Fourier Representation: Example

$$u(x,y) = \sum_{k_y=-k_{\max}}^{k_{\max}} \sum_{k_x=-k_{\max}}^{k_{\max}} \hat{u}(k_x,k_y)\exp[i(k_x x + k_y y)]$$

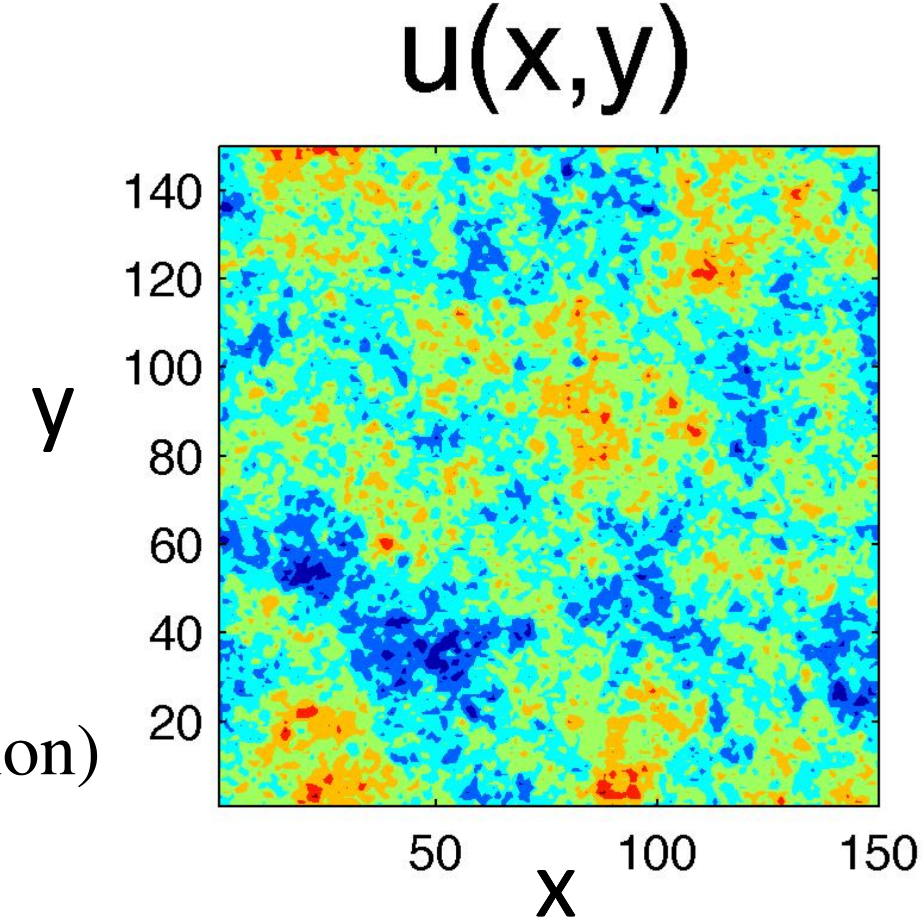
$$k_{\max} = \pi N / L$$

$$N = 150 \text{ modes}$$

$$\hat{u}(k_x,k_y) = a(k)\exp[i\phi]$$

$$a(k) \sim 1/k$$

ϕ : Random (uniform distribution)

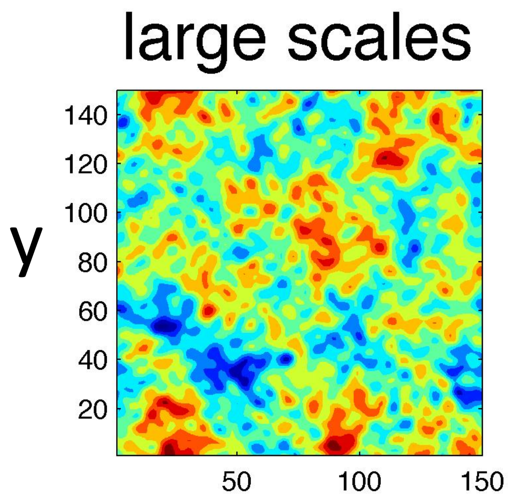


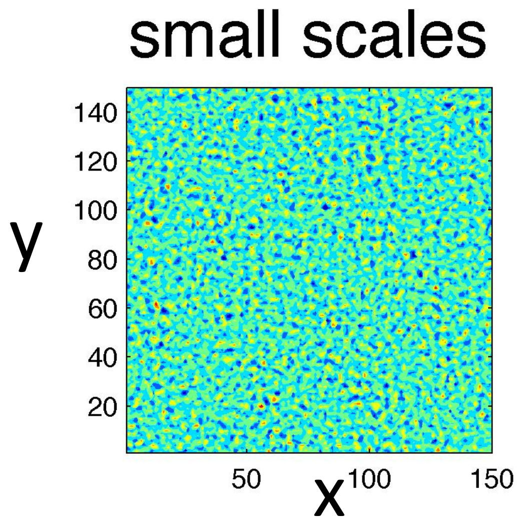
2D Fourier Representation: Example

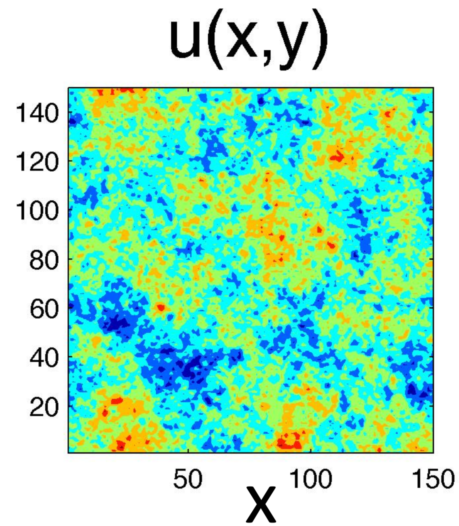
$$u^{\text{Large}}(x,y) = \sum_{k_y=-\frac{k_{\max}}{8}}^{\frac{k_{\max}}{8}} \sum_{k_x=-\frac{k_{\max}}{8}}^{\frac{k_{\max}}{8}} \hat{u}(k_x,k_y)\exp[i(k_x x + k_y y)]$$

$$u^{\text{Small}}(x,y) = u(x,y) - u^{\text{Large}}(x,y)$$

Large scales contain the “features”







Meaning of Energy Spectra

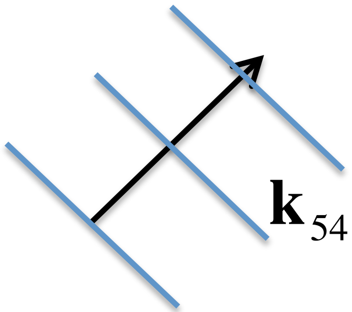
❑In 2D:

$$\frac{1}{2}\overline{u_i u_i} = \frac{1}{2L^3}\int u_i u_i \mathrm{d}\mathbf{x} = \frac{1}{2}\sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k})\cdot\hat{\mathbf{u}}^*(\mathbf{k}) = \sum_{m=-N/2}^{N/2-1}\sum_{n=-N/2}^{N/2-1}\hat{E}(k_x^m, k_y^n)$$

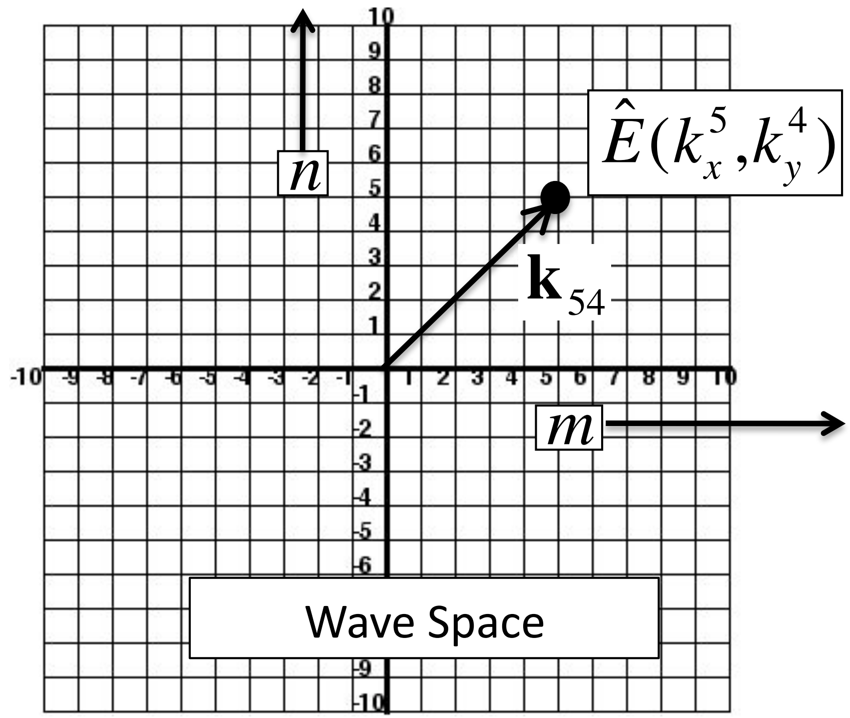
$$\hat{E}(k_x^m, k_y^n) = \frac{1}{2}\hat{\mathbf{u}}(\mathbf{k}_{mn})\cdot\hat{\mathbf{u}}^*(\mathbf{k}_{mn}),$$

$$\mathbf{k}_{mn} = k_x^m\mathbf{i} + k_y^n\mathbf{j},$$

$$k_x^m = \frac{2\pi m}{L}, k_x^n = \frac{2\pi n}{L}$$



Physical Space



Wave Space

Meaning of Energy Spectra

- Energy Spectra:

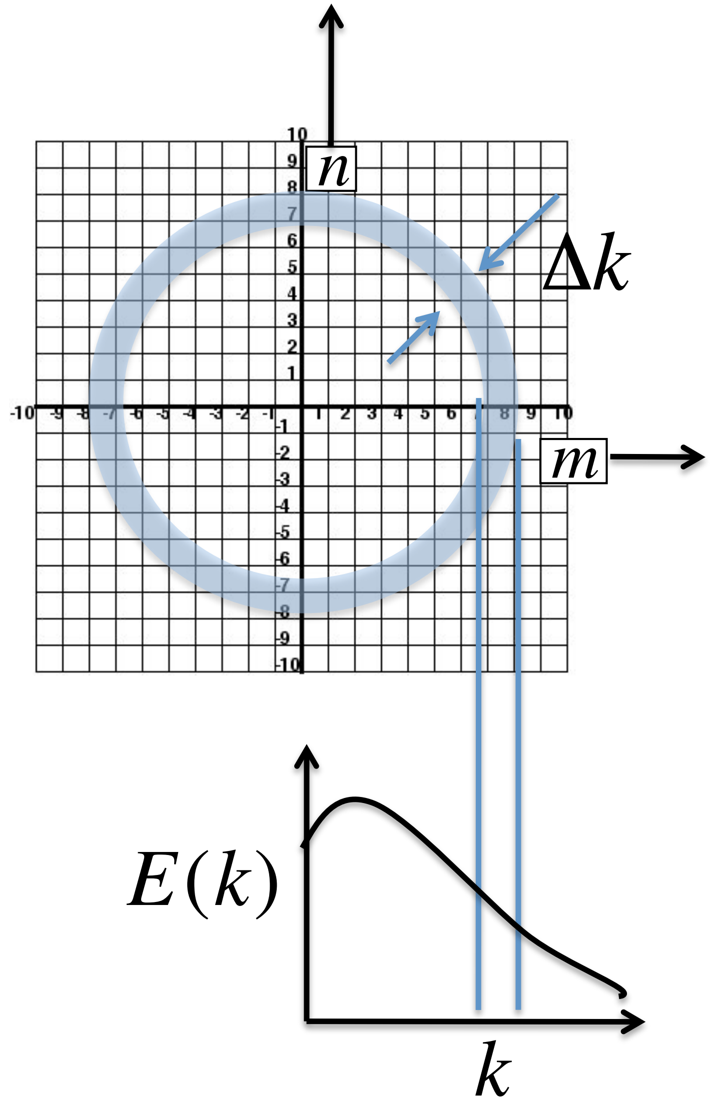
$$E(k_i) = \frac{1}{\Delta k} \sum_{\mathbf{k};(i-1)\Delta k \le k < i\Delta k} \hat{E}(k_x^m, k_y^n)$$

$$\frac{1}{2}\overline{u_i u_i} = \frac{1}{2L^3}\int \overline{u_i u_i}\,\mathrm{d}\mathbf{x} = \sum_{i=1}^{M} E(k_i)\Delta k$$

In general $M \neq N$

- In limit of $M \to \infty$

$$\frac{1}{2}\overline{u_i u_i} = \int_0^\infty E(k)dk$$



Meaning of Energy Spectra

- We want:

$$\overline{u^2} = \int_0^\infty E(k)dk$$

- Turbulent Kinetic Energy is:

$$K = \frac{1}{2}\overline{u_i(\mathbf{x})u_i(\mathbf{x})} \quad = \quad \frac{1}{2}\sum_{\boldsymbol{\kappa}} \overline{\hat{u}_i(\boldsymbol{\kappa})\hat{u}_i^*(\boldsymbol{\kappa})}$$

- To convert summation to integral,

$$E(\kappa) = \frac{1}{2}\sum_{\boldsymbol{\kappa}'} \overline{\hat{u}_i(\boldsymbol{\kappa}')\hat{u}_i^*(\boldsymbol{\kappa}')}\delta(\kappa - |\boldsymbol{\kappa}'|)$$

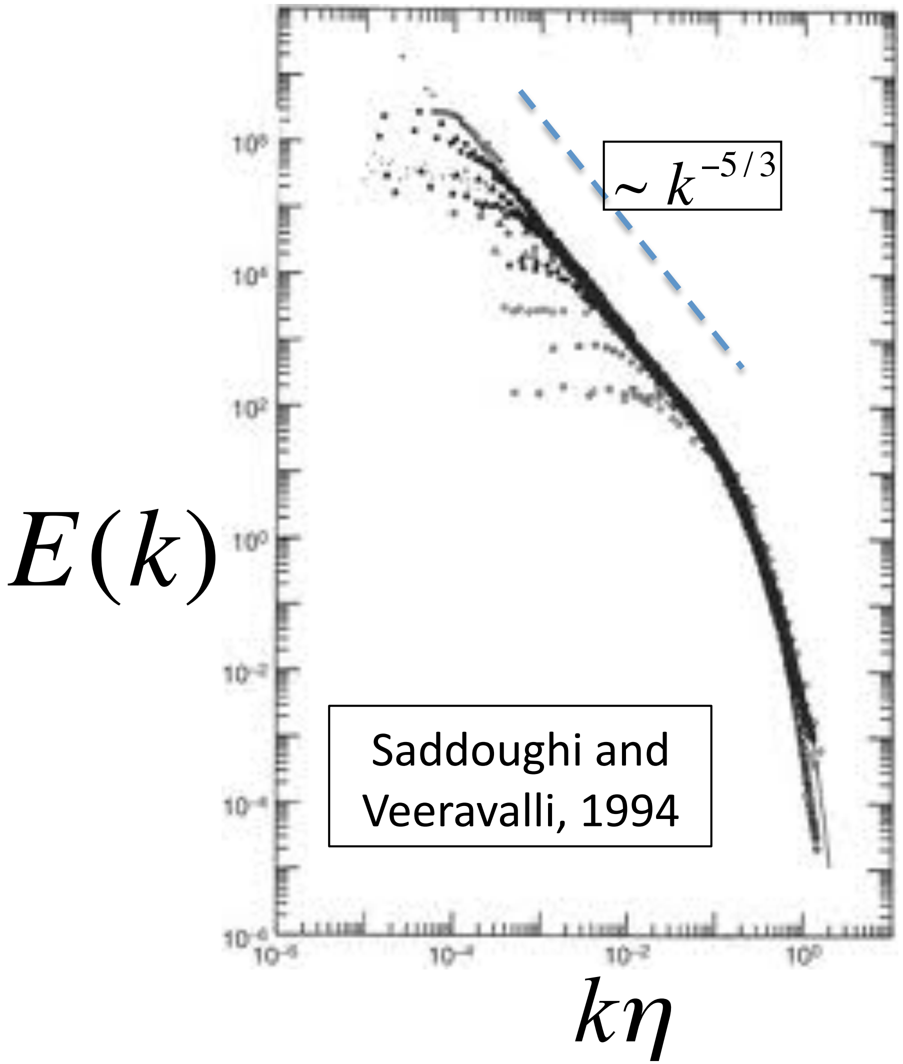
Kolmogorov's Energy Spectra

- For a 3D flow field:

Experimentally:

$$\boxed{E(k) \sim k^{-5/3}}$$

Power law form can be explained by scale-wise energy balance



Energy Balance For Forced Turbulence

- Stirring can be considered as forcing at small wavenumbers (large scales)

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \bullet \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \underbrace{\mathbf{f}(\mathbf{x},t)}_{\text{Stirring with Spoon}}$$

$$\mathbf{f}(\mathbf{x},t) = \int_{k<k_f} \hat{\mathbf{f}}(\mathbf{k},t) d\mathbf{k}; \;\; k_f \sim \frac{2\pi}{L_{spoon}}$$

- Eqn for average turbulent kinetic energy

$$K = \frac{1}{2}\overline{\mathbf{u} \bullet \mathbf{u}} \qquad \frac{\partial K}{\partial t} = -\underbrace{\nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}}_{\varepsilon\,=\text{Dissipation Rate}} + \underbrace{\overline{f_k u_k}}_{\text{Power Input}}$$

Energy Balance For Forced Turbulence

- Start with

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_k}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} + f_i$$

- Multiply by velocity and average..

$$\overline{u_i \frac{\partial u_i}{\partial t}} + \overline{u_i \frac{\partial u_i u_k}{\partial x_k}} = -\overline{u_i \frac{\partial p}{\partial x_i}} + \overline{\nu u_i \frac{\partial^2 u_i}{\partial x_k \partial x_k}} + \overline{u_i f_i}$$

- Use continuity, statistical homogeniety:

$$\frac{\partial K}{\partial t} = -\underbrace{\nu \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}}}_{\varepsilon \,=\, \text{Dissipation Rate}} + \underbrace{\overline{f_k u_k}}_{\text{Power Input}}$$

Energy Balance For Forced Turbulence

- After statistical stationarity is reached (stirring for a long time)

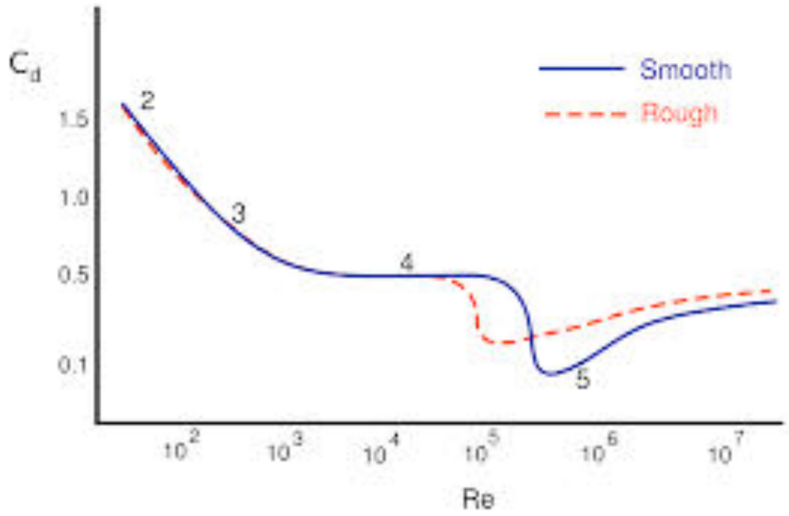
$$\frac{\partial K}{\partial t} = 0 \Rightarrow \underbrace{\nu \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}}}_{\varepsilon = \text{Dissipation Rate}} = \underbrace{\overline{f_k u_k}}_{\text{Power Input}}$$

- **Question:** I am stirring the spoon with velocity *U* and force *F*. Will *F* go up or down if I decrease viscosity (for a fixed *U*) ?
  - Let's look at experimental data…

Drag Law For Sphere

- For a sphere of radius *R:*

$$F = \frac{1}{2} C_D \rho \pi R^2 U^2$$



- C_D(Re)=constant
  - Re= UR/ν
- *Dissipation rate does not vary with viscosity !*

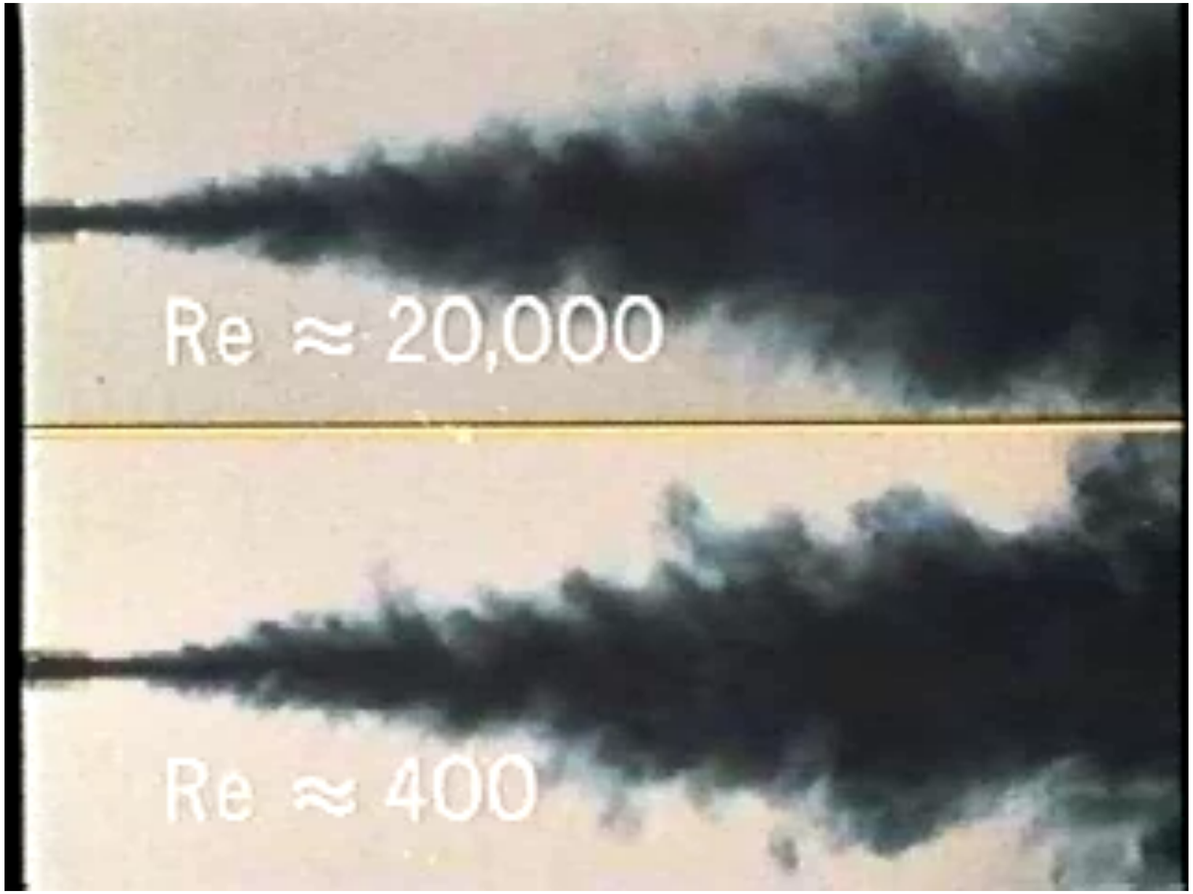
$$\varepsilon = \text{Dissipation per unit mass} = \frac{FU}{\rho R^3} = \frac{1}{2} C_D \pi \frac{U^3}{R}$$

Eddies Are Smaller at Higher Re

$$\varepsilon = \nu \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}} = \text{Constant}$$

$$\Rightarrow \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}} \text{ increases with Re}$$

1. Larger range of length scales at higher Reynolds numbers
2. Large fraction of dissipation rate occurs in small scale



Dissipation Spectra

$$\frac{\partial u_i}{\partial x_j}(\mathbf{x}) \;=\; \sum_{\boldsymbol{\kappa}} i k_j \hat{u}_i(\boldsymbol{\kappa}) \exp[i\boldsymbol{\kappa}\cdot\mathbf{x}]$$

which leads to

$$\begin{aligned}
\nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}(\mathbf{x}) &= \sum_{\boldsymbol{\kappa}} \overline{-ik_j\hat{u}_i(\boldsymbol{\kappa}) ik_j \hat{u}_i(-\boldsymbol{\kappa})} \\
&= \sum_{\boldsymbol{\kappa}} k^2 \overline{\hat{u}_i(\boldsymbol{\kappa})\hat{u}_i(-\boldsymbol{\kappa})}
\end{aligned}$$

$$\hat{\epsilon}(\kappa) \;=\; \sum_{\boldsymbol{\kappa}'} {\kappa'}^2 \overline{\hat{u}_i(\boldsymbol{\kappa}')\hat{u}_i^*(\boldsymbol{\kappa}')}\delta(\kappa - |\boldsymbol{\kappa}'|)$$

$$\hat{\epsilon}(\kappa) \;=\; \kappa^2 \sum_{\boldsymbol{\kappa}'} \overline{\hat{u}_i(\boldsymbol{\kappa}')\hat{u}_i^*(\boldsymbol{\kappa}')}\delta(\kappa - |\boldsymbol{\kappa}'|) = \kappa^2 E(\kappa)$$

Dissipation Occurs at Higher Wavenumbers (small scales)

- Definition of dissipation spectra:

$$\varepsilon = \int \hat{\varepsilon}(k)dk$$

- It can be shown that:

$$\hat{\varepsilon}(k) = k^2 E(k)$$

- If $E(k) \sim k^{-n}$ and $n<2$ then dissipation rate is higher for higher k
  - Typically n=-5/3 $\Rightarrow \hat{\varepsilon}(k) = k^{1/3}$
- Dissipation takes place in small scales !

So far, we have understood:

- Energy is generated at large scales
- Energy is dissipated at smallest scales
- Convective terms in N.S. equation do not contribute to global energy balance
  - They can only transfer energy between scales
- Flow with higher Re will have a larger range of scales

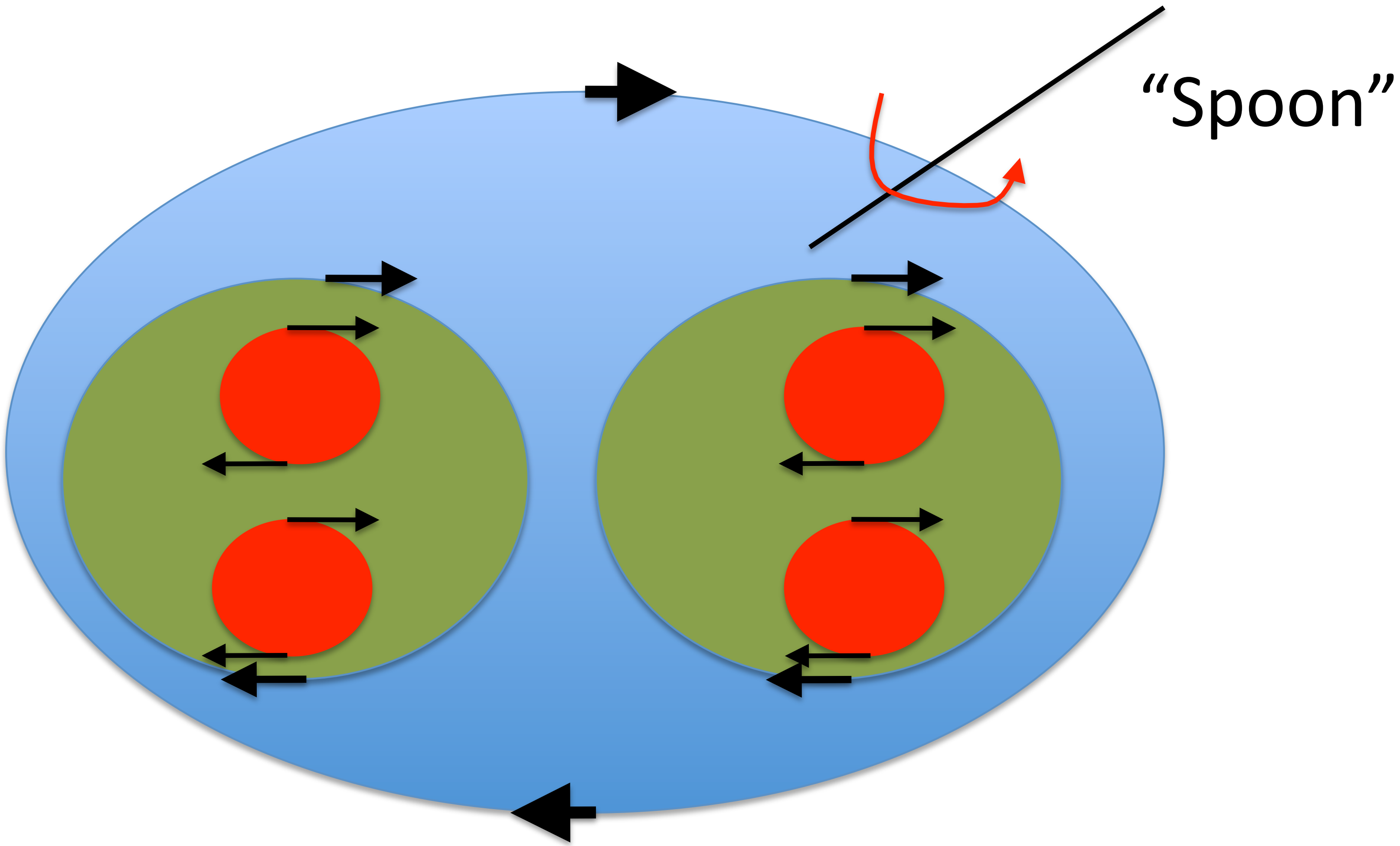
We can now understand Kolmogorov's 1941 Theory

Cascade of Energy In Turbulence
Richardson's Poem

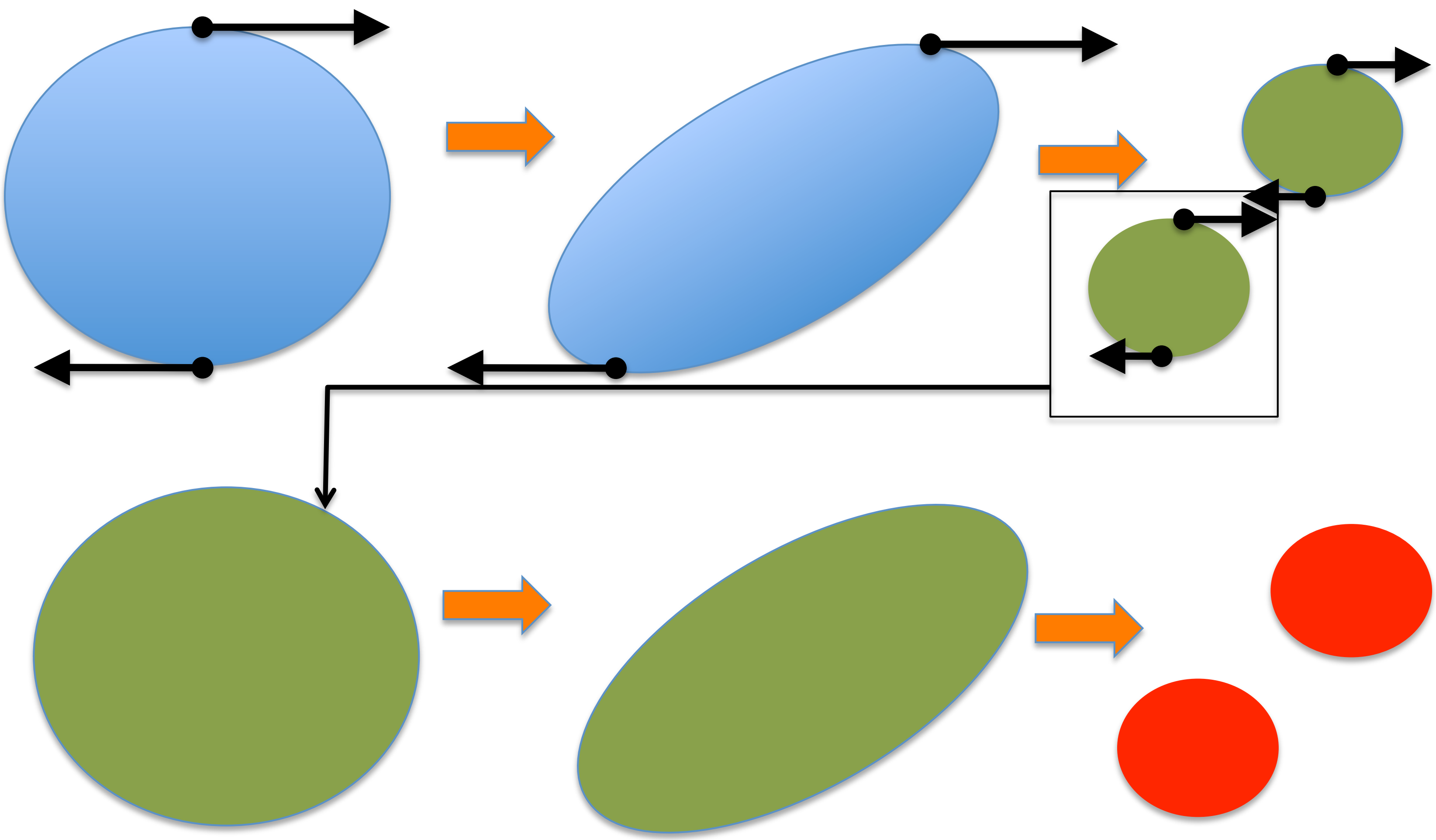
Big whorls have little whorls, little whorls have smaller whorls, that feed on their velocity, and so on to viscosity.

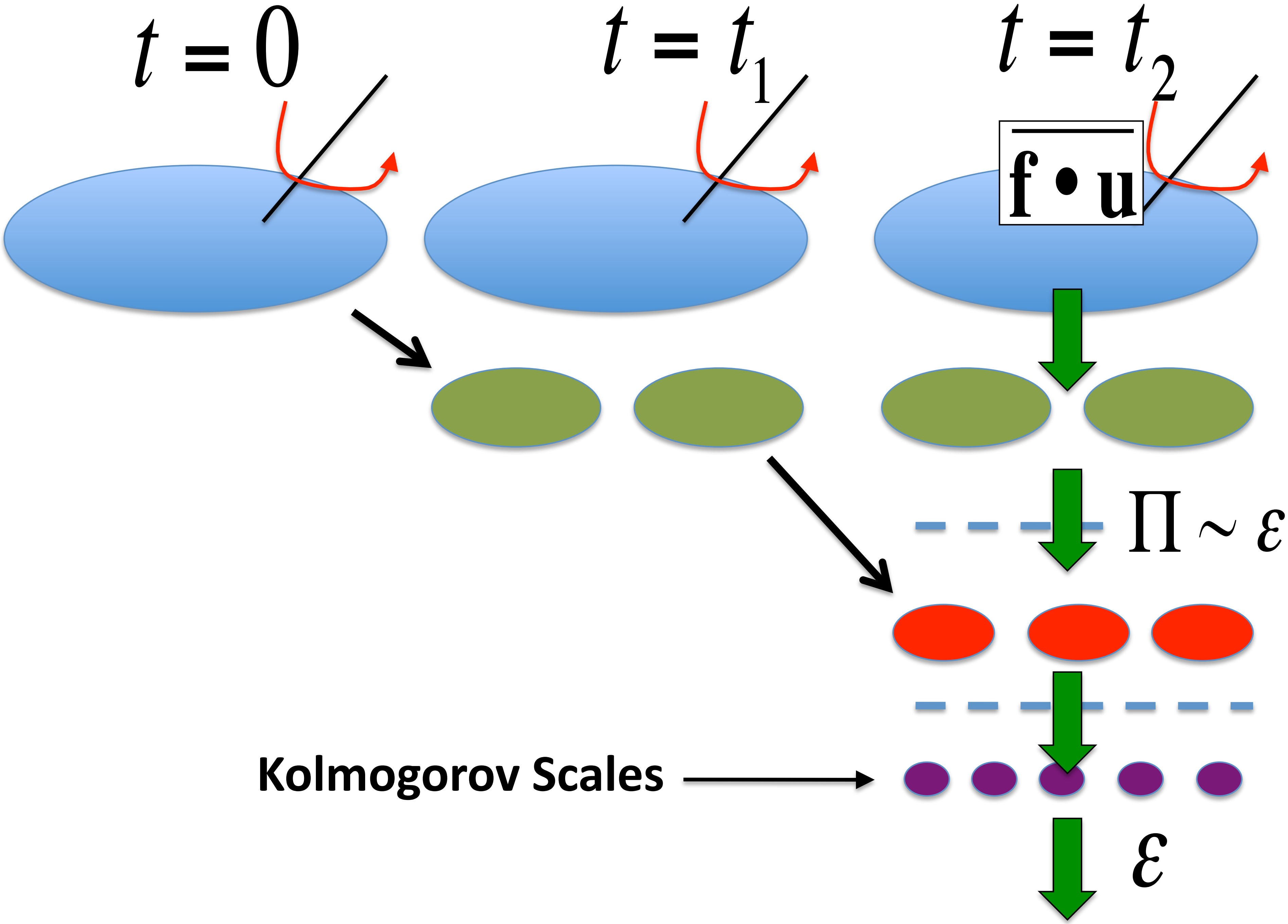
Lewis F. Richardson (1922)

Turbulent "Eddy" Distribution



Self-Similar Energy Cascade





$t = 0$
$t = t_1$
$t = t_2$
$\overline{\mathbf{f} \cdot \mathbf{u}}$
$\Pi \sim \varepsilon$
Kolmogorov Scales
ε

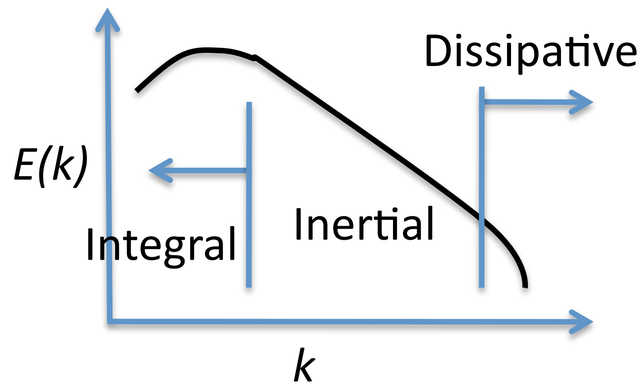
Length Scales of Flow

- Integral Scale

$$L = K^{3/2}/\varepsilon$$

- Dissipative Scale (Kolmogorov Scale)

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$



- Inertial Scales

$$L >> \lambda >> \eta$$

5/3rds Law for *E(k)*

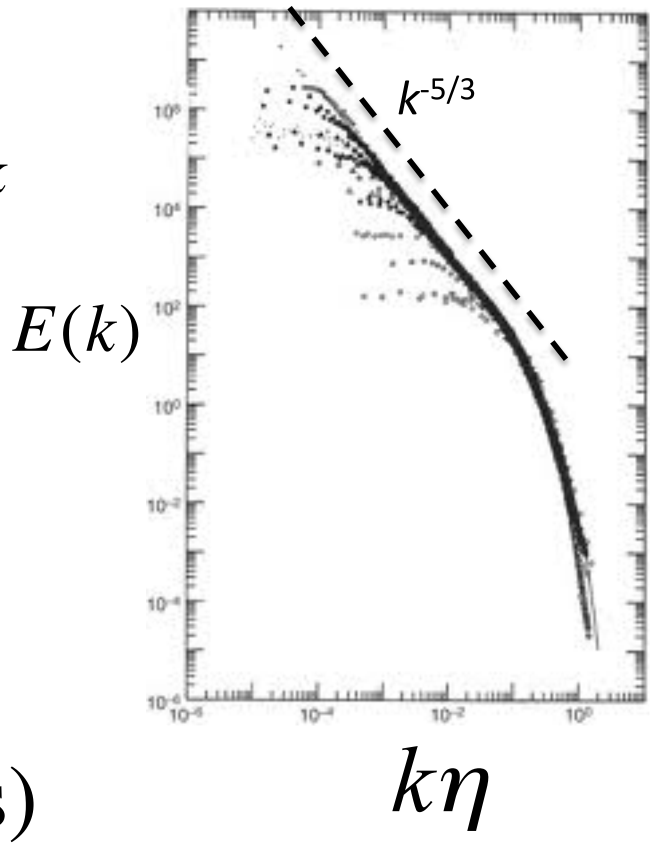
- In the inertial range, only energy transfer from large to small scale takes place
- Only relevant quantities are: ε, k
- Therefore, we postulate:

$$E(k) = F_E(\varepsilon, k)$$

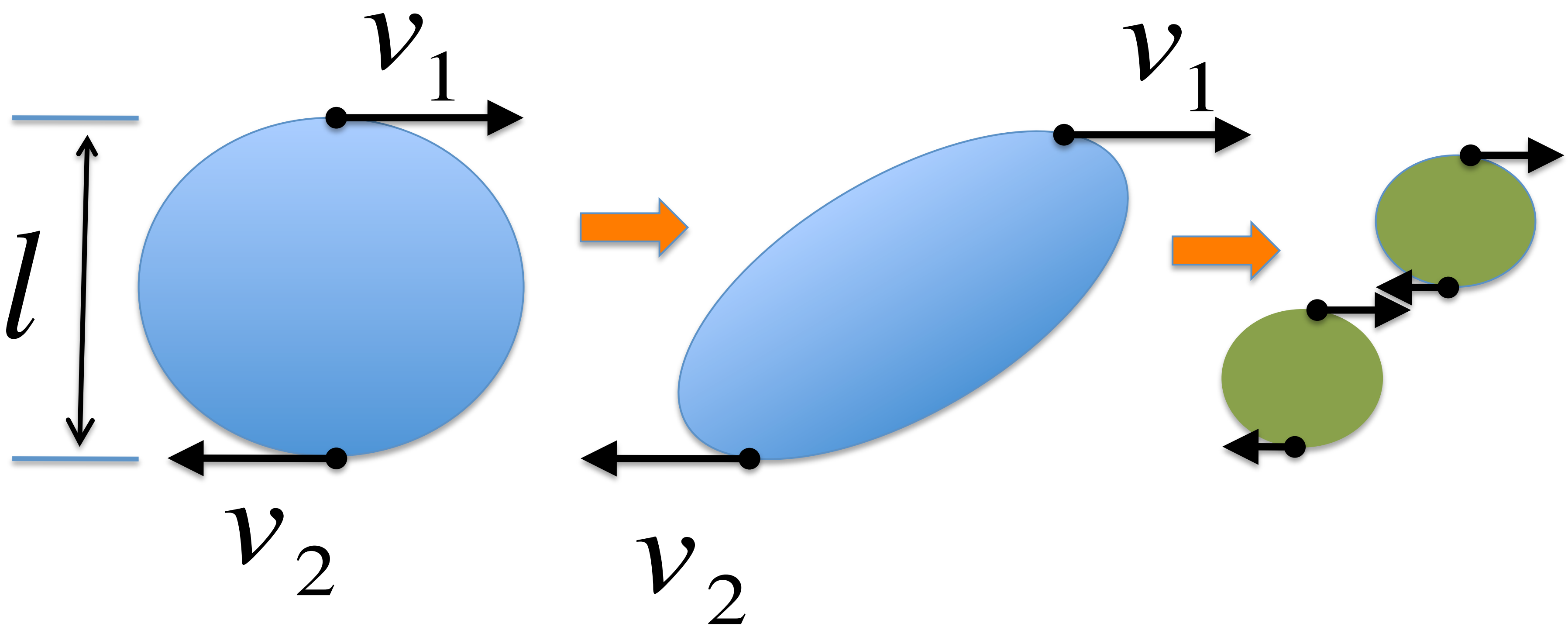
- Using dimensional analysis:

$$E(k) = C_K \varepsilon^{2/3} k^{-5/3}$$

$$C_K \sim 1.5 \text{ (Experiments)}$$



Time-scales in Energy Cascade



$$v_l = |v_1 - v_2| \Rightarrow t_l \sim l/v_l$$

$$\text{Energy transfer rate} = \Pi_l \sim \frac{v_l^2}{t_l} = \frac{v_l^3}{l} = \varepsilon \Rightarrow t_l \sim l^{2/3}\varepsilon^{-1/3}$$

Physical Meaning of Kolmogorov Scale

- For any eddy with length-scale l, viscous time scale is:

$$t_\nu = \frac{l^2}{\nu}$$

- Inertial time scale:

$$t_l = \frac{l^{2/3}}{\varepsilon^{1/3}}$$

- Equating viscous, inertial time scales ($t_l = t_\nu$):

$$l = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} = \eta \quad \text{(Kolmogorov length scale)}$$

Two-point Correlation vs Spectra

- 2-point correlation (in 1D): $R(r) = \overline{u(x)u(x+r)}$

$$\boxed{R(r)} = \frac{1}{L}\int_0^L u(x)u(x+r)\mathrm{d}x = \frac{1}{L}\sum_{k,k'}\hat{u}(k)\hat{u}(k')\int_0^L \exp[i\{kx + k'(x+r)\}]\mathrm{d}x$$

$$= \sum_{k,k'}\hat{u}(k)\hat{u}(k')\exp[ik'r]\frac{1}{L}\int_0^L \exp[i(k+k')x]\mathrm{d}x$$

$$= \sum_{k,k'}\hat{u}(k)\hat{u}(k')\exp[ik'r]\delta_{k,-k'} = \boxed{\sum_{k}\hat{u}(k)\hat{u}^*(k)\exp[ikr]}$$

$R(r)$ and $|\hat{u}(k)|^2$ are Fourier transform pairs !

Structure Function vs Spectra

- "Structure function" in 1-D:

$$\boxed{\Delta R(r)} = R(0) - R(r) = \sum_k \left|\hat{u}(k)^2\right| [1 - \exp(ikr)]$$

$$\boxed{= \sum_{k \geq 0} \left|\hat{u}(k)^2\right| [1 - \cos(kr)]}$$

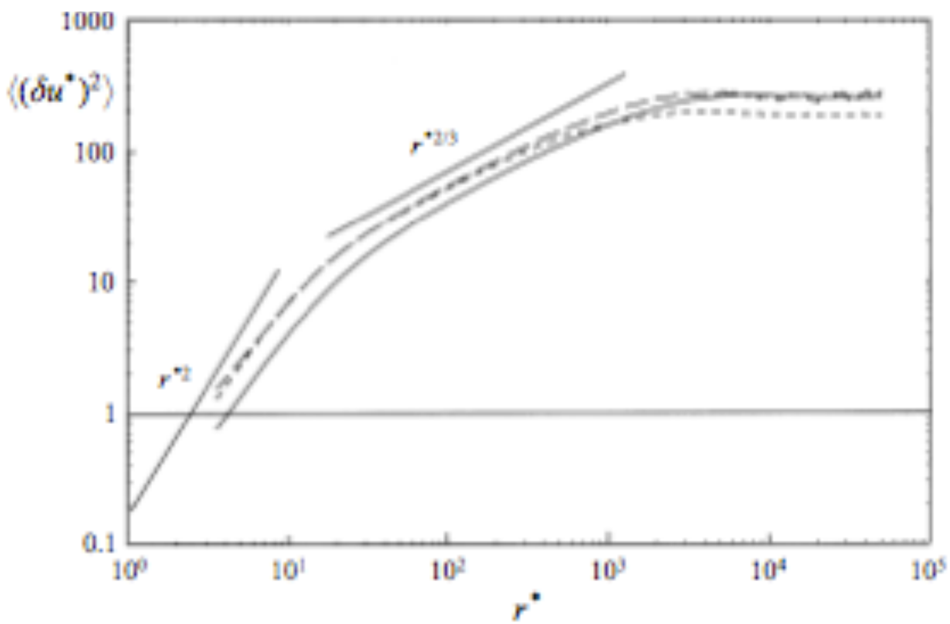
$$\text{For } kr << 1, \quad \cos(kr) - 1 \approx 0$$

- *ΔR(r)* "filters" out energy of scales much larger than *r*

Longitudinal Structure Function

- For a 3D flow field $\mathbf{u}(\mathbf{x})$

$$S_2^L(r) = \overline{\left[u_1(\mathbf{x}+\mathbf{i}\,r) - u_1(\mathbf{x})\right]^2} = \overline{\delta u_1(r)^2}$$
$$= 2\left[\overline{u_1(\mathbf{x})u_1(\mathbf{x})} - \overline{u_1(\mathbf{x}+\mathbf{i}\,r)u_1(\mathbf{x})}\right]$$



Experimentally:

$$\boxed{S_2^L \sim r^{2/3}}$$

2/3rds Law for $S_2(r)$

- In the inertial range, only energy transfer from large to small scale takes place
- Only relevant quantities are: ε, r
- Therefore, we postulate:

$$S_2^L(k) = F_S(\varepsilon, r)$$

- Using dimensional analysis:

$$S_2^L(r) = C\varepsilon^{2/3} r^{2/3}$$

$$C = 2.0 \text{ (Experiments)}$$

Cost of Direct Numerical Simulation (DNS)

- Navier Stokes equation is solved with high resolution in space and time
  - …and therefore high computational cost

$$\text{Kolmogorov Scale}: \ \eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}, \ \text{Large length scale}: \ L = \frac{k^{3/2}}{\varepsilon},$$

$$\Rightarrow \text{No. of grid pts in each dirn}: \ N \sim \frac{L}{\eta} = \mathrm{Re}_T^{3/4}, \ \text{Turb Reynolds number}: \ \mathrm{Re}_T = \frac{k^2}{\nu\varepsilon}$$

$$\text{Time step}: \ \Delta t = CFL\frac{L}{k^{1/2}N}, \ \text{Eddy turnover time}: \ T \sim \frac{k}{\varepsilon}$$

$$\Rightarrow \text{No. of time steps}: \ N_t \sim \frac{T}{\Delta t} = \mathrm{Re}_T^{3/4}$$

$$\text{Computational cost (isotropic turbulence)}: \ N^3 N_t \sim \mathrm{Re}_T^3$$