ME 704 : 2014 : Quiz 2: 80 min — Maximum Marks = 20

1. THREE Formulae sheets (A4 size, unruled) are allowed. All sheets should be in your own handwriting.
2. EACH formula sheet should have a 1 cm margin on the top with your name and roll number written clearly in pen.
3. NO photocopies. Any photocopy implies ZERO marks in the exam.
4. NO Text-book; NO Print-outs of any kind.

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1. Consider the 1-D, first order, linear hyperbolic equation

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}, a > 0$$

Now consider the following scheme for its solution

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{a}{2}\left[\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} + \frac{u_{i+1}^{n} - u_{i-1}^{n}}{2\Delta x}\right]$$

Use the Von Neumann method for stability analysis to find a stability criterion for the scheme. In this method, do not bother about movement along a particular direction.

(10 marks)

2. Consider the equation

$$Y'' + 4Y' + 13Y = 40cos(x)$$

$4 \times 4 + 13 \times 3 = 40$

$1 - 16$
$= -15$

with initial condition given as $Y(0) = 3$ and $Y'(0) = 4$. Convert this into a set of first order differential equations. Solve this set using the Euler method for a step size of h = 0.05 and 0.1. Solve for 4 steps using h = 0.05 and for 2 steps using h = 0.1 . Calculate the error at x=0.2 by comparing both the solutions with the exact solution

$$Y(x) = 3cos(x) + sin(x) + e^{-2x}sin(3x)$$

(10 marks)

$5 - 13 \times 3.4$
$+ 40 \times \cos(1)$

$y_2' = -4y_2 - 13y_1 + 40\cos x$

$\rightarrow y_{11} = y_{10} + h(y_{10}') = 3 + 0.1 \times 4 = \underline{3.4}$

$y_{21} = y_{20} + h(y_{20}') = 4 + 0.1 \times (-16) = \underline{4 - 1.5} = 2.5$

ME 704 : 2015 : MidSem Quiz : 120 min Maximum Marks = 30

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1. Consider the set of equations with the unknowns x_1, and x_2:

$$x_1 + 4x_2 = 1$$

$$4x_1 + x_2 = 0$$

   (a) Beginning with the starting guess $\{x_1, x_2\} = \{0,0\}$, show five iterations using the Jacobi and Gauss-Siedel methods. (4 marks)

   (b) Next interchange the two equations, i.e.,

$$4x_1 + x_2 = 0$$

$$x_1 + 4x_2 = 1$$

   Use the same two methods with the same starting guess and show five iterations. (4 marks)

   (c) What is the reason for the difference in approaches 1 and 2. In each case, how is the Gauss-Siedel performing with respect to Jacobi? (2 marks)

2. Consider the function: $y = sin^2(x) \times cos(x)$.

   (a) Create a table of {x,y} by evaluating the function in the interval at the points: 0.0, 0.25, 0.5, 0.75, 1.0 (i.e. steps of 0.25). Note that the values are in radians. There are 5 points in all. (1 mark)

   (b) Using Trapezoidal rule, find the integral of this function between 0 and 1 (using the above set of points). (2 marks)

   (c) Find the exact value using analytical integration and hence find the error. (2 marks)

   (d) What is the estimated error? Hence, calculate the value of the integral using modified Trapezoidal rule. (2 marks)

   (e) In the formula for the exact error, at what point should you calculate the necessary derivative? Show how you will obtain this using methods described in class. (3 marks)

3. Small questions

(a) Consider the following set of points: $(x_1, y_1) = (1, 4.1)$, $(x_2, y_2) = (4.5, 11.45)$, $(x_3, y_3) = (8, 18.8)$.
Pass a second order polynomial $(ax^2 + bx + c)$ exactly through these three points. Use Lagrange's method. Find the values of the polynomial coefficients. **(5 marks)**

(b) Consider the curve given by the equation: $y = 5x^3 + 2x^2 + 3x + 1$, in the interval 0 to 1. If a straight line is drawn between the points $\{0, y(0)\}$ and $\{1, y(1)\}$, what is the point in the interval where the slope of the curve matches the slope of the line?

(2 marks)

(c) Assume that you want to evaluate $\int_a^b f(x)g(x)dx$ where $f(x)$ can be any function and $g(x)$ is always $\frac{1}{\sqrt{x}}$. If a single point evaluation using Gaussian Integration is to be carried out where the interval from $\{a, b\}$ is mapped to $\{0, 1\}$, then where should the function be evaluated and what should be the value of the weight? This formula should be exact when $f(x)$ is a polynomial of order 1 or below.

(3 marks)

ME 704 : 2015 : Quiz 3 : 75 min Maximum Marks = 20

1. One Formula sheet allowed (A4 size sheet: both sides). Formula sheet sheet should be in your own handwriting.
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3. NO Text-book; NO Print-outs of any kind.

Consider the 1-D Parabolic Equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2},$$

where α is a positive constant.
Consider the following method for solving u_i at time step $(n+1)$:

$$u_i^{n+1} = \frac{2d}{1+2d} u_{i+1}^n + \frac{1-2d}{1+2d} u_i^{n-1} + \frac{2d}{1+2d} u_{i-1}^n$$

Here d is short hand notation for:

$$\frac{\alpha(\Delta t)}{(\Delta x)^2}$$

Note that the method involves time at three levels : $n+1$, n and $n-1$.

1. Use **Von-Neumann** stability analysis to evaluate the stability criterion for the method.

   (13 marks)

2. Use **Taylor Series** analysis to check whether this method is consistent with the equation. What is the order of the method? (7 marks)

ME 704 : 2015 : Quiz 2 : 75 min — Maximum Marks = 20

1. One Formula sheet allowed (A4 size sheet: both sides). Formula sheet sheet should be in your own handwriting.
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1. Consider the equation

$$Y'' + 4Y' + 13Y = 40cos(x)$$

with initial condition given as $Y(0) = 3$ and $Y'(0) = 4$. Convert this into a set of first order differential equations. Solve this set using the Euler method for a step size of h = 0.05 and 0.1. Solve for 4 steps using h = 0.05 and for 2 steps using h = 0.1 . Calculate the error at x=0.2 by comparing both the solutions with the exact solution

$$Y(x) = 3cos(x) + sin(x) + e^{-2x} sin(3x)$$

(10 marks)

2. Consider the equation

$$\frac{dy}{dx} = xy,$$

with initial condition given as $y_0 = e$ when $x_0 = 0$. Use the 2nd order AB method with $h = 0.1$ to obtain the values of y at $x = 0.1, 0.2, 0.3,$ 0.4 and 0.5. To begin, you can use RK2 with $\gamma_2 = 0.5$ (i.e., Heun's method). Tabulate the values using the Numerical scheme, the exact values (using integration) and the error. Also find the value at x = 0.5 using RK2 directly with $h = 0.5$. (10 marks)

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1. Consider the equation f(x) = $(1+x^2)\cos(x)$ in the interval 0.5 to 1.5. For evaluation of the cos function imagine that "x" is in radians.

   (a) Use the trapezoidal rule with 4 intervals to obtain the value of the integral. Also use the modified trapezoidal rule to get a better answer.

   (5 marks)

   (b) Use Gaussian Numerical integration (Note the limits of integration) with 4 function evaluations. The values of x_is are +/- 0.8611363116 (both with w_i= 0.3478548451) and +/- 0.3399810436 (both with w_i= 0.6521451549). (5 marks)

2. Consider the set of equations with the unknowns x_1, and x_2:

$$x_1 + 4x_2 = 1$$

$$4x_1 + x_2 = 0$$

   (a) Beginning with the starting guess $\{x_1, x_2\} = \{0,0\}$, show five iterations using the Jacobi and Gauss-Siedel methods. Which method diverges faster? (5 marks)

   (b) Next interchange the two equations, i.e.,

$$4x_1 + x_2 = 0$$

$$x_1 + 4x_2 = 1$$

   Use the same two methods with the same starting guess and show five iterations. Which method converges faster? (5 marks)

3. Consider the following set of points: (x_1, y_1) = (1, 4.1), (x_2, y_2) = (4.5, 11.45), (x_3, y_3) = (8, 18.8),

   (a) Pass a second order polynomial (ax^2 + bx + c)exactly through these three points. Use Lagrange's method. Find the values of the polynomial coefficients. (5 marks)

   (b) Assume you want to fit a quadratic polynomial ' ax^2 + bx + c' through the 3 points using a least square fit method. Find the coefficients a, b, c. (5 marks)

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1. Open Notes exam. All notes should be in your own handwriting.
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1. (a) Consider the curve given by the equation: $y = 5x^3 + 2x^2 + 3x + 1$, in the interval 0 to 1. If a straight line is drawn between the points {0, y(0)} and {1, y(1)}, what is the point in the interval where the slope of the curve matches the slope of the line? (2 marks)

   (b) Consider the equation: $y = e^x$; in the interval 0 to 1. Expanding about x=0, you are finding the value of the function at x=1 using a Taylor series expansion. If you incorporate terms up to that including x^4, find the value of "x" in the interval which will ensure that the error term (i.e. the Taylor residue) is exact. Assume that the value of the function given by your calculator is correct up to 8 decimal places. (2 marks)

2. Consider the set of equations with the unknowns x_1, x_2 and x_3:

$$2x_1 + 6x_2 + 3x_3 = -3.6$$

$$5x_1 + x_2 + 7x_3 = 11.5$$

$$10x_1 + 4x_2 + 3x_3 = 9$$

   (a) Beginning with the starting guess $\{x_1, x_2, x_3\} = \{0,0,0\}$, show five iterations using the Jacobi Method. (4 marks)

   (b) Using any method (known to you from before), find the exact solution for the above. (2 marks)

   (c) Comment on the behaviour of the Jacobi iterates. (2 marks)

3. Consider the function: y = cos(x). Create a table of {x,y} by evaluating the function in the interval at the points: 0.0, 0.1... (in steps of 0.1)..., 0.9,1.0. (Note that the values are in radians). There are 11 points in all. (1 mark)

   (a) Using Simpson's rule find the integral of this function between 0 and 1 (using the above set of points). Find the exact value using analytical integration and hence find the error. Compare this error with the estimated error using the expression given in class. (4 marks)

   (b) Assume you want to fit a quadratic polynomial ' $ax^2 + bx + c$' through the 11 points using a least square fit method. Find the coefficients a, b, c. (4 marks)

4. Write a pseudocode for the following (add comments and make it understandable). First clearly think about the logic and then write the codes. Too many scratches and over-writing will imply that the answer will be given zero marks.

(a) Assume you are solving a 1-D problem. You are given an array X[NMAX] where the temperature is known and given by the array T[NMAX]. NMAX is the total number of points in the array. At all these points the first and second derivatives have to be calculated and stored in the arrays FD[NMAX] and SD[NMAX] respectively. Both are calculated using first order central difference schemes for the inner points. For the left-most point, both derivatives are calculated using forward difference (first order) and for the right-most point, they are calculated using backward difference (first order). **(4.5 marks)**

(b) Assume there is a horizontal beam and there is a load acting vertically on it. The load is acting at all points. However, it is known only at points X[NMAX] where NMAX is the total number of points in the array. The values of the force are stored in the array F[NMAX]. You have to get the net load acting on the beam by integrating the force using the Trapezoidal rule.

(4.5 marks)

$\underline{M_1} = \underline{M_n} = 0$

$\frac{1}{6} M_1 + \frac{2}{3} M_2 + \frac{1}{6} M_3 = \left(\frac{1}{3} - \frac{1}{2}\right)$

$1 M_1 + 0 M_2 + 0 M_3 + 0 M_4 = 0$

$\frac{1}{6}$

$0 M_1 + 0 M_2 + 0 M_3 + M_4 = 0$

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1. **Stability Analysis:** Consider the 1-D Parabolic Equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2},$$

where α is a positive constant.
Use **Von-Neumann** stability analysis to evaluate the stability criterion for the following method:

$$(1+2d)u_i^{n+1} = (1-2d)u_i^{n-1} + 2d[u_{i+1}^n + u_{i-1}^n]$$

$U^n e^{jxi}$

Here d is short hand notation for:

dk

$$\frac{\alpha(\Delta t)}{(\Delta x)^2}$$

$U^{n+1}_{i+1} = U^{n+1} e^{jx(i+1)}$

(10 marks)

2. Consider the 1-D wave equation

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x},$$

where a is a positive constant.

Consider the situation with a = 50 m/s, $\Delta x = 10^{-2}$ m, $\Delta t = 10^{-4}$ s. The total domain is divided into 100 points (i.e., length is 1 metre). If the individual points are labelled as "i", the initial conditions (at t=0) can be written $u_i = 0$ for $0 < i \leq 4$; $u_i = 10$ for $5 \leq i \leq 7$; $u_i = 0$ for $i \geq 8$. The boundary conditions are u(0,t) = u(100,t) = 0. Time-steps are represented using the variable n, starting with n=0 representing t=0. Use the following method (with $c = a\Delta t/\Delta x$):

$$u_i^{n+1} = u_i^n - \quad [u_{i+1}^n - u_{i-1}^n] + \frac{1}{2}c^2[u_{i+1}^n - 2u_i^n + u_{i-1}^n]$$

to take 3 time steps (n=1, n=2, n=3). Make a neat plot of the solution at n = 0, 1, 2 and 3. Show points i = 2, 3, 4, 5, 6, 7, 8, 9 and 10 clearly each time. How is the solution behaving. Is it as expected? If the behaviour is correct, explain why this method should give proper solutions. If the solution has problems, explain what kind of problems exist and why.

(10 marks)

3. **Calculations and Number Crunching** Consider the system of equations (so-called Predator-Prey model, the derivative can be assumed to be w.r.t. time):

$$Y_1' = AY_1[1 - BY_2]; Y_1(0) = Y_{1,0}$$

and,

$$Y_2' = CY_2[DY_1 - 1]; Y_2(0) = Y_{2,0}$$

Consider the case with $A = 4$, $B = 0.5$, $C = 3$, $D = 0.5$, $Y_{1,0} = 3$, $Y_{2,0} = 1$.

Consider the simplest RK2 formulation (where symbols have their usual meaning):

$$y_{j,n+1} = y_{j,n} + \frac{h}{2}\Big[f_j(x_n, y_{1,n}, y_{2,n}) +$$

$$f_j(x_{n+1}, y_{1,n} + hf_1(x_n, y_{1,n}, y_{2,n}), y_{2,n} + hf_2(x_n, y_{1,n}, y_{2,n})\Big]$$

Take 3 step sizes of h = 0.2 each, and plot Y_1 and Y_2 (on the same graph) as a function of time.

Similarly, take 3 steps of h = 0.2 each and plot both quantities again if all other quantities are kept constant, but initial conditions are changed to: $Y_{1,0} = 1$, $Y_{2,0} = 3$.

Comment on the behaviour of the solution. Which is the predator and which is the prey?

(10 marks)

4. (a) The Lax method for solving the linear 1-D wave equation ($\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}$, where a is a positive constant) is given as:

$$u_i^{n+1} = \frac{1}{2}[u_{i+1}^n + u_{i-1}^n] - \frac{a\Delta t}{2\Delta x}[u_{i+1}^n - u_{i-1}^n].$$

$x + 4\beta - 72 + 25$

-18

Does the scheme satisfy the sufficient condition for a TVD scheme? Please demonstrate clearly one way or the other.

(5 marks)

(b) Define s(x) as
$x^3 + 2x^2 + 1$; for $1 \le x \le 2$ and,
$-2x^3 + \beta x^2 - 36x + 25$; for $2 \le x \le 3$.
Find the value of β for which s(x) is a cubic spline function on [1,3]. Verify that it is indeed a cubic spline function on this interval. Is it a natural cubic spline function on this interval?

(5 marks)

1 2 3

$\boxed{f(x) = f(x)}$

$\boxed{f'(2) = f'(x}$

$Y_1^* = X$

$Y_2^* = Y$

$6 \cdot x + 4 = -2 \times 3 \times 2x + 2\beta$

$-24 + 2\beta$

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1. (a) Consider the equation: $y = e^x$; in the interval 0 to 1. Expanding about x=0 using Taylor series, you are told to find the value of the function at x=1. If you incorporate terms up to that including x^4, find the value of "x" in the interval which will ensure that the error term (i.e. the Taylor residue) is exact. Assume that the value of the function given by your calculator is correct up to 8 decimal places. **(2 marks)**

   (b) Suppose you want to find the square root of a real number "a". Show that the Newton scheme for solving this can be written as

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

   Show also that the error formula can be written as

$$\sqrt{a} - x_{n+1} = -\frac{1}{2x_n}(\sqrt{a} - x_n)^2$$

   (3 marks)

2. Consider the set of equations with the unknowns x_1, x_2 and x_3:

$$2x_1 + 6x_2 + 3x_3 = -3.6$$

$$5x_1 + x_2 + 7x_3 = 11.5$$

$$10x_1 + 4x_2 + 3x_3 = 9$$

   (a) Keeping the system intact, begin with the starting guess $\{x_1, x_2, x_3\}$ = {0,0,0}, and show four iterations using the Jacobi Method. **(3 marks)**

   (b) Using substitution/Gauss elimination/Determinants or any other method you know of, find the exact solution for the above. **(2 marks)**

   (c) Comment on the reason for the behaviour of the Jacobi iterates. **(1 mark)**

   (d) How will you re-arrange the system to make the iterative scheme work better? Show 4 iterations with the re-arranged system and the same initial guess as before. **(4 marks)**

3. Consider the function: $y = x \times cos(x)$.

(a) Create a table of {x,y} by evaluating the function in the interval at the points: 0.0, 0.25, 0.5, 0.75, 1.0 (i.e. steps of 0.25). Note that the values are in radians. There are 5 points in all. (1 mark)

(b) Using Trapezoidal rule, find the integral of this function between 0 and 1 (using the above set of points). (2 marks)

(c) Re-calculate the integral using the modified trapezoidal rule. (1 mark)

(d) Using Simpson's rule find the integral of this function between 0 and 1 (using the above set of points). What is the estimated error? (2 marks)

(e) Find the exact value using analytical integration and hence find the error using each of the above methods. (2 marks)

(f) Assume you want to fit a quadratic polynomial ' $ax^2 + bx + c$' through the 5 points using a least square fit method. Find the coefficients a, b, c. (3 marks)

4. Write a pseudocode for the following (add comments and make it understandable). First clearly think about the logic and then write the code. Too many scratches and over-writing will imply that the answer will be given zero marks.

Assume you are solving a 1-D problem. You are given two arrays of size NMAX (total number of points):
i) X[NMAX] : co-ordinates of the points and,
ii)T[NMAX] : temperature at these points.
At all these points the first and second derivatives of the temperature field have to be calculated and stored in the arrays FD[NMAX] and SD[NMAX] respectively. Both are calculated using first order central difference schemes for the inner points. For the left-most point, both derivatives are calculated using forward difference (first order) and for the right-most point, they are calculated using backward difference (first order).
Your code should

(a) Read the value of NMAX

(b) Read the values of X and T arrays.

(c) Calculate the first and second derivatives using a loop and appropriate IF conditions.

(d) Write a NEAT OUTPUT with the X value of a point, its temperature and the respective first and second derivatives on each line.

(4 marks)