ME673 – Endsem (Nov 15th 2015, 10:00 am, Duration: 3 hours Total points: 50)

Instructor: Amitabh Bhattacharya

Instructions Please read the following instructions carefully

- This QP contains FOUR printed sides with SEVEN questions – please check.
- This is NOT an open book exam.
- NO cheat sheet is allowed; all the necessary information has been provided in the question paper itself.
- Put your mobiles and bags away.
- Calculators are allowed.
- Justify each step in your solution as far as possible. Points may be deducted for not stating assumptions/steps.
- Please number the pages and indicate page number for each question on the front page of answer book.
- If you are attempting a question in different parts of the answer book, then please write "continued on page X" at the end of each part.

Questions:

1. (4 points) Consider the equation $x[xy']' + \lambda^2 y = 0$ for $y(x)$, with boundary condition $y(a) = y(b) = 0$, where a, b are positive real numbers. Find the eigenvalue and eigenfunctions which satisfy this equation for the above boundary value problem and derive appropriate orthogonality relations for the eigenfunctions.

2. (4 points) Consider Legendre's equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, where n is an integer. Using this equation, find the form of $P_2(x)$, the Legendre polynomial of order 2, upto a constant factor. Points will not be given for simply writing down the final answer here. (Hint: You need to take power series around $x = 0$).

3. (8 points) A spring-mass system consisting of two masses located at $x_1(t)$ and $x_2(t)$ satisfies the following coupled 2nd order equations at any point in time t:

$$\cancel{\ddot{x}_1 + 2\dot{x}_1 - x_2} = \sin t \qquad \ddot{x}_1 + 2x_1 - x_2 = \sin t$$

$$\ddot{x}_2 - x_1 + 2x_2 = -\sin t$$

Find $x_1(t)$ and $x_2(t)$ for $t > 0$, given that $x_1(0) = \dot{x}_2(0) = 0$ and $\dot{x}_1(0) = x_2(0) = 1$

4. (8 points) Consider the differential equations for $f(r)$ and $g(r)$:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right] g(r) = 0 \qquad (1)$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right] f(r) = g(r) \qquad (2)$$

Solve for $f(r)$ using the boundary values $f(1) = 1$, $f'(1) = 1$, along with the condition that f and f' are both finite at $r = 0$.

5. (8 points) Consider a semi-infinite string, extending in space from $0 \le x \le \infty$. The displacement of the string, $y(x,t)$, is governed by the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ for time $t \ge 0$. The initial conditions are $y(x,0) = 0$ and $\frac{\partial y}{\partial t}(x,0) = \delta(x-1)$, where $\delta(x)$ is the Dirac delta function. The boundary condition is $y(0,t) = 1 - \exp(-t)$ for $t \ge 0$ ($y(0,t) = 0$ for $t < 0$). Find the solution for $y(x,t)$ for $x \ge 0$ and $t \ge 0$.

6. (10 points) Temperature $T(r,\theta,t)$ in a 2D disc of radius a satisfies the following homogeneous diffusion equation:

$$\frac{\partial T}{\partial t} = \alpha^2 \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right]$$

where $0 \le r \le a$, $0 \le \theta \le 2\pi$, (r,θ) being the polar coordinates of any point in space, $t > 0$ being time. The following boundary and initial conditions are applied on the disc: $T(2\pi + \theta) = T(\theta)$ $\qquad T(0, \theta, t) = \text{finite}$

$$T(a,\theta,t) = t\sin\theta; \quad T(r,\theta,0) = r\sin\theta \rightarrow \text{IC}$$

Find the expression for $T(r,\theta,t)$ for $t > 0$, $0 \le r \le a$, and $0 \le \theta \le 2\pi$, $\cancel{(r,\theta)}$. (Note: Boundary condition $T(a,\theta,t)$ is dependent on time; this is not a typographic error.)

$\theta \rightarrow$ $\qquad$ $r \rightarrow$

$T(2\pi + \theta)$ $\qquad$ $T(a, \theta, t) = t\sin\theta$

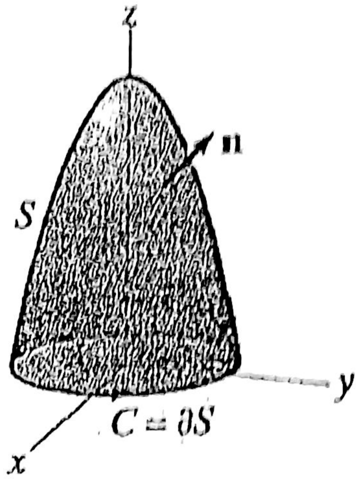

Figure 1: Figure for problem 7

7. (8 points) Let S be the paraboloid parametrized in terms of (r, θ) as $S = \{(x, y, z); x = r\cos\theta, y = r\sin\theta, z = 9 - r^2; 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$. The boundary of the surface, ∂S, is then simply the circle given by $\partial S = \{(x, y, z); x = 3\cos\theta, y = 3\sin\theta, z = 0\}$. Given a field $\mathbf{F}(\mathbf{x}) = (2z - y)\mathbf{i} + (x + z)\mathbf{j} + (3x - 2y)\mathbf{k}$, verify the Stokes theorem, $\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \oint_{\partial S} \mathbf{F} \cdot \mathbf{ds}$ for this field.

Useful Mathematical Formulas

- Bessel's Equation is:

$$x^2 y'' + xy' + (\lambda x^2 - \mu^2) y = 0$$

with solution $J_{\pm\mu}(\lambda x)$ if μ is a non-integer.

- Orthogonality of Bessel's functions:

$$\int_0^1 J_m(\gamma_{m,n} r) J_m(\gamma_{m,n'} r) r\,dr = \begin{cases} 0 & \forall n \neq n' \\ \frac{1}{2} J_{m+1}^2(\gamma_{mn}) & \forall n = n' \end{cases}$$

where γ_{mn} is the n^{th} root of $J_m(r)$.

ME673: Quiz 3 (30 mins)

Total points: 10. Justify each step in your solution as far as possible. Quiz is closed book. Please keep away your mobile (after switching off) and bags. Only calculator is allowed.

1. (10=6+2+2 points) Consider the following ODE:

$$(x^4 + 2x^2)y'' + 3xy' - 6x^2y = 0$$

(a) Construct a power series solution around $x = 0$, and find the general recursion relation relation between successive coefficients in the series.

(b) Find the first two *non-zero* terms of the power series for the two linearly independent solutions.

(c) What is the radius of convergence for the power series solutions ?

ME673: Quiz 2 (30 mins)

Total points: 10. Justify each step in your solution as far as possible. Quiz is closed book. Please keep away your mobile (after switching off) and bags. Only calculator is allowed.

1. (6=5+1 points) Consider the following system of ODEs:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

where $\mathbf{x}(t) \in \mathbb{R}^3$, and $\mathbf{A}$, which is a 3×3 symmetric matrix, has the following eigenvalues and eigenvectors:

$$\begin{aligned} \lambda_1 = -1, \quad \mathbf{e}_1 &= [-1, 2, 1]^T \\ \lambda_2 = 2, \quad \mathbf{e}_2 &= [-1, -1, 1]^T \\ \lambda_2 = 7, \quad \mathbf{e}_3 &= [1, 0, 1]^T \end{aligned}$$

   (a) Find the solution $\mathbf{x}(t)$ for $0 \leq t < \infty$ if $\mathbf{x}(0) = [1, 0, 0]^T$.

   (b) What is the maximum possible value of the condition number κ for $\mathbf{A}^2$?

2. (4 points) The matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

can be expressed as $\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1}$, where $\mathbf{Q}$, $\mathbf{D}$ are 3×3 matrices and $\mathbf{D}$ is a Diagonal matrix. Find $\mathbf{Q}$ and $\mathbf{D}$.

ME673 – Midsem

(Sept 10th 2015, 5:30 pm, Duration: 2 hours Total points: 50)

Instructor: Amitabh Bhattacharya (Dept. Mech. Engg., I.I.T.B.)

Note: This is NOT an open book exam. NO cheat sheet is allowed. Only calculator is allowed. Put your mobiles and bags away. SWITCH OFF YOUR MOBILES BEFORE YOU PUT THEM AWAY. Justify each step in your solution as far as possible.

1. (4 points) What is the range in x over which following power series converges ?

$$\sum_{n=1}^{\infty} \frac{n^{50}}{n!}(x+7)^n$$

2. (5 points) Use Lioville's formula for the Wronskian, $W(t) = W(\xi)\exp\left[-\int_{\xi}^{t} p_1(t')dt'\right]$, to show whether or not $y_1(t) = t^3$ and $y_2(t) = t^4$ can both be solutions to $y'' + p_1(t)y' + p_2(t)y = 0$ for $t \in \mathbb{R}$, where $p_1(t)$ and $p_2(t)$ are continuous over all $t \in \mathbb{R}$. Here $W(t)$ is the Wronskian based on y_1 and y_2. State your reason clearly – just mentioning the final answer will not fetch any points.

3. (5 points) Find the general solution for (x, y, z), which satisfies these equations:

$$\begin{aligned} x + 4y + 2z &= 0 \\ 2x + y + 5z &= 0 \end{aligned}$$

What is the basis and dimension of the solution space ?

4. (12=6+6) Consider the following second order ODE for $y(t)$:

$$4ty'' + 2y' - y = 4\sqrt{t}\exp\sqrt{t}$$

(a) Use the transformation $z = \sqrt{t}$ to find the general solution to the homogeneous version of the above equation.

(b) Find the particular solution of Eqn (1) (in original or transformed version), using the variation of parameters. Points will NOT be given if you use any other method here.

You may use the identity $\int x\exp(ax)dx = \exp(ax)(ax-1)/a^2$

5. (12=6+6 points) Consider the following equation for $x(t)$:

$$m\ddot{x} + c\dot{x} + kx = \sin(\Omega t)$$

with initial conditions $x(0) = \dot{x}(0) = 0$. Find the solutions for $x(t)$, given:

(a) $m = 1$, $c = 1$, $k = 1$, and $\Omega = 1$

(b) $m = 1$, $k = 1$, $c = 0$, $\Omega = 1$. Draw a labeled graph of $x(t)$ vs t for this case.

6. (12=3+5+4 points) Consider the following coupled first order ODEs:

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{f}(t) \tag{1}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} -5t+2 \\ -8t-8 \end{bmatrix} \tag{2}$$

(a) Find the general solution for $\mathbf{x}^h(t)$ which satisfies the homogeneous version of the above equation,

$$\frac{d\mathbf{x}^h}{dt} = \mathbf{A}\mathbf{x}^h \tag{3}$$

(b) Suppose $\mathbf{x}_1(t) = [x_1(t) \quad y_1(t)]^T$ and $\mathbf{x}_2(t) = [x_2(t) \quad y_2(t)]^T$ are two linear independent solutions to the homogeneous equation given by Eqn (3). We can try to construct the particular solution $\mathbf{x}_p(t)$ (which therefore has to satisfy Eqn (1)) using variation of parameters as follows:

$$\mathbf{x}_p(t) = v_1(t)\mathbf{x}_1(t) + v_2(t)\mathbf{x}_2(t)$$

where $v_1(t)$ and $v_2(t)$ are scalar functions of t. What are the UNCOUPLED first order ODEs we need to solve for $v_1(t)$ and $v_2(t)$, given $x_1(t)$, $y_1(t)$, $x_2(t)$, $y_2(t)$?

(c) Using the method suggested above derive (but DO NOT solve) the uncoupled ODEs for $v_1(t)$ and $v_2(t)$ for the SPECIFIC expressions of $\mathbf{A}$ and $\mathbf{f}(t)$ given in Eqn (2).

ME673: Quiz 4 (30 mins)

Total points: 10. Justify each step in your solution as far as possible. Quiz is closed book. Please keep away your mobile (after switching off) and bags. Only calculator is allowed.

1. (4 points) Verify the Divergence Theorem for the vector field $\mathbf{v}(r,\theta,z) = 3r^2\mathbf{e}_r - r\mathbf{e}_\theta + 2\mathbf{e}_z$, in which the closed region is given by the cylinder $r \leq 4$, $0 \leq \theta \leq 2\pi$, $0 \leq z \leq 5$. Here (r,θ,z) are cylindrical coordinates, and $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ are the unit vectors along the coordinate axes. In Cylindrical coordinates, for $\mathbf{v}(r,\theta,z) = v_r(\mathbf{r})\mathbf{e}_r + v_\theta(\mathbf{r})\mathbf{e}_\theta + v_z(\mathbf{r})\mathbf{e}_z$, divergence is given by $\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial v_r r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$.

2. (6 = 2+2+2 points) A surface S is parametrized by u and v, so that $S = \{(x,y,z); x = u,\ y = u^2,\ z = v, \text{ where } 0 < u < 1,\ 0 < v < 1\}$. You are also given a scalar function $F(x,y,z) = xz^2$.

   (a) What is the differential area of the surface, dS, corresponding to the differential increment in the parameters du and dv ? You need to DERIVE the expression for dS; points will *not* be given for *just* stating the formula.

   (b) Find the surface integral $\int_S F(\mathbf{x})dS$. Make sure you apply the limits for u and v given above.

   (c) Find the expression for $\mathbf{n}$, the unit normal on the surface, at a particular value of (u,v). (Hint: The cross product of two vectors tangent to the surface will generate a vector normal to the surface.)

1. 03 sheets of formulae are allowed. Each sheet should be A4 size.
2. All sheets should be in your own handwriting.
3. NO photocopies of any material (including notes).
4. NO Text-book; NO Print-outs of any kind.

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1. **Stability Analysis:** Consider the 1-D wave equation

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x},$$

where a is a positive constant.
Consider the following schemes (where c is used to denote $a\frac{\Delta t}{\Delta x}$) and find the stability criterion for each of the schemes using the method specified.

(a)

$$u_i^{n+1} = \frac{c}{2}u_{i-1}^n + u_i^n - \frac{c}{2}u_{i+1}^n$$

Use the **Discrete Perturbation** method. **(7 marks)**

(b)

$$u_i^{n+1} = \frac{c(c+1)}{2}u_{i-1}^n + (1-c^2)u_i^n + \frac{c(c-1)}{2}u_{i+1}^n$$

Use **Von-Neumann** stability analysis. **(8 marks)**

2. Consider the 1-D wave equation in problem 1 and consider the following scheme for solving it (where c is again short-hand notation for $a\frac{\Delta t}{\Delta x}$) :

$$u_i^{n+1} = \frac{(1+c)}{2}u_{i-1}^n + \frac{(1-c)}{2}u_{i+1}^n$$

The individual points are labelled as "i" as per the scheme and time-steps are represented using the variable n, starting with n=0 representing t=0.

(a) Expand each term which is not u_i^n as a Taylor series and use this to determine the order of this scheme. **(4 marks)**

(b) Is the solution scheme consistent with the equation? (The above analysis should help you). **(2 marks)**

(c) Does the scheme satisfy the sufficient condition for being a TVD scheme? **(4 marks)**

(d) Now consider you are solving the given equation with the given scheme on a domain of 1 metre. The domain is divided into 100 points, i.e. $\Delta x = 10^{-2}$ m. The value of a is 100 m/s, and $\Delta t = 10^{-5}$ s. The initial conditions (at t=0) can be written as $u_i = 0$ for $0 < i \leq 5$; $u_i = 10, 20, 10$ for $i = 6, 7, 8$ respectively; $u_i = 0$ for $i \geq 9$. The boundary conditions are u(0,t) = u(100,t) = 0. Take 3 time steps (n=1, n=2, n=3). Make a neat plot of the solution at n = 0, 1, 2 and 3. Show points i = 3, 4, 5, 6, 7, 8, 9, 10 and 11 clearly each time. How is the solution behaving. Is it as expected? Can it be made better by changing Δt. If so, suggest a Δt that will make the solution as good as possible and take one step with this Δt to show the behaviour.

(10 marks)

3. **Calculations and Number Crunching**

(a) Consider the system of equations (so-called Predator-Prey model, the derivative can be assumed to be w.r.t. time):

$$Y_1' = AY_1[1 - BY_2]; Y_1(0) = Y_{1,0}$$

and,

$$Y_2' = CY_2[DY_1 - 1]; Y_2(0) = Y_{2,0}$$

Consider the case with $A = 4$, $B = 0.5$, $C = 3$, $D = 0.5$, $Y_{1,0} = 3$, $Y_{2,0} = 1$.

Consider the simplest RK2 formulation (where symbols have their usual meaning):

$$y_{j,n+1} = y_{j,n} + \frac{h}{2}\Big[f_j(x_n, y_{1,n}, y_{2,n}) +$$

$$f_j(x_{n+1}, y_{1,n} + hf_1(x_n, y_{1,n}, y_{2,n}), y_{2,n} + hf_2(x_n, y_{1,n}, y_{2,n}))\Big]$$

Take 3 step sizes of h = 0.2 each, and plot Y_1 and Y_2 (on the same graph) as a function of time.

Similarly, take 3 steps of h = 0.2 each and plot both quantities again if all other quantities are kept constant, but initial conditions are changed to: $Y_{1,0} = 1$, $Y_{2,0} = 3$.

Comment on the behaviour of the solution. Which is the predator and which is the prey? **(9 marks)**

(b) Let α be the smallest positive root of the equation:

$$1 - x + sin(x) = 0$$

Find an interval $[a, b]$ containing α for which the bisection method will converge to α. Justify the choice of this interval. For your interval, estimate the number of iterates needed to find α to an accuracy of 1×10^{-8}. Show 5 iterations. **(6 marks)**